must be locally Euclidean and itself a one-parameter group. Then $G^*$
determines a one-parameter subgroup of $G$. Suppose that $k > 1$ and that
the theorem is true for $n < k$. If $G$ has dimension $k$ and is locally con-
connected, then we can find a one-parameter subgroup of the subgroup
guaranteed by Lemma 6, whose dimension is positive and less than $k$.
Finally suppose that $G$ has dimension $k$ but is not locally connected.
Then $G^*$ is a locally compact, locally connected group of positive dimension
at most $k$, so $G^*$ contains a one-parameter subgroup, which in turn gives
us a one-parameter subgroup of $G$.

2. Montgomery, D., "Theorems on the Topological Structure of Locally Compact
(1947).

PSEUDO-CONFORMAL GEOMETRY OF POLYGENIC FUNCTIONS
OF SEVERAL COMPLEX VARIABLES

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1. A complex function

$$w = F(z^a) = F(z^1, \ldots, z^n) = \phi(x^a; y^a) + i\psi(x^a; y^a),$$

where the real $\phi$ and $\psi$ are single-valued continuous functions and possess
continuous partial derivatives over a region $R$ of $2n$ dimensional real space
$\Sigma_z$ of coordinates $(x^1, \ldots, x^n; y^1, \ldots, y^n)$ is termed a polygenic function
of the $n$ complex variables $z^a = x^a + iy^a$, where $a = 1, \ldots, n$.

We shall study the first derivatives of a polygenic function and also pre-
sent some new results in pseudo-conformal geometry.

2. For the class of polygenic functions, the linear operators

$$\frac{\partial}{\partial z^a} = \frac{1}{2} \left( \frac{\partial}{\partial x^a} - i \frac{\partial}{\partial y^a} \right); \quad \frac{\partial}{\partial s^a} = \frac{1}{2} \left( \frac{\partial}{\partial x^a} + i \frac{\partial}{\partial y^a} \right),$$

are important. These are called the mean and phase derivatives, respecti-
vely. The operations $\frac{\partial}{\partial s^a}$ and $\frac{\partial}{\partial z^a}$ are not partial derivatives but signify
the application of the linear operators (2),
3. A polygenic function \( w \) is monogenic over \( R \) if and only if

\[
\frac{\partial w}{\partial z^\alpha} = 0, \quad \text{where } \alpha = 1, \ldots, n. \tag{3}
\]

These are equivalent to the \( 2n \) Cauchy-Riemann equations. The real and imaginary parts obey the \( n^2 \) Poincaré partial differential equations of second order

\[
\frac{\partial^2 w}{\partial z^\alpha \partial z^\beta} = 0, \quad \text{where } \alpha, \beta = 1, \ldots, n. \tag{4}
\]

A multiharmonic function is any solution \( w \) of this Poincaré system.\(^3\) If \( w = \phi + i\psi \) is monogenic over \( R \), then \( \phi \) and \( \psi \) are conjugate-mutiharmonic over \( R \).

4. The locus of points in \( \Sigma_{2m} \), defined by the equations

\[
z^\alpha = f^\alpha(Z^\beta), \tag{5}
\]

where the \( nf^\alpha \) are monogenic functions over a region of the \( 2m \) dimensional parametric space \( \Sigma_{2m} \) of the \( m \) complex variables \( Z^\beta = X^\beta + iY^\beta \), where \( \beta = 1, \ldots, m \), such that the jacobian-matrix

\[
\begin{pmatrix}
\frac{\partial x^\alpha}{\partial X^\beta} & \frac{\partial x^\alpha}{\partial Y^\gamma} \\
\frac{\partial y^\alpha}{\partial X^\beta} & \frac{\partial y^\alpha}{\partial Y^\gamma}
\end{pmatrix}, \tag{6}
\]

is of rank \( 2m \), is called a pseudo-conformal manifold \( \Sigma_{2m} \) of \( 2m \) dimensions. Evidently \( m \leq n \).

If \( m = n \), there results a correspondence, called a pseudo-conformal transformation \( T \), between the two pseudo-conformal manifolds \( \Sigma_{2m} \) and \( \Sigma_{2n} \). All such maps \( T \) form the pseudo-conformal group \( G \). The associated geometry is termed pseudo-conformal geometry.

Under \( G \), any \( \Sigma_{2m} \) becomes a \( \Sigma_{2m} \). For \( m = 1 \), any \( \Sigma_{2} \) is called a conformal or analytic surface. The reason for this terminology is that \( G \) induces conformality between pairs of analytic surfaces.

The angle \( \lambda \), between any direction at a fixed point \( P \) on a conformal surface \( S \) and its orthogonal projection on the \( s^a \)-plane depends only on the position of the point \( P \) on \( S \). Moreover the sum of the squares of the cosines of the resulting \( n \) angles \( \lambda_1, \ldots, \lambda_n \), at \( P \), is unity.\(^3\)

An area \( A \) on a conformal surface \( S \) is the sum of the projected areas \( A_\alpha \) on the \( s^\alpha \)-planes.

5. Consider a polygenic function \( w \) defined over a conformal surface \( S \) in \( R \). If \( Z = X + iY \) is the parameter describing \( S \), then

\[
\frac{dw}{dZ} = \left( \frac{\partial w}{\partial x^\alpha} \frac{dz^\alpha}{dZ} \right) + \left( \frac{\partial w}{\partial z^\alpha} \frac{dz^\alpha}{dZ} \right)e^{2i\theta}, \tag{7}
\]
in which the repeated index $\alpha$ means to sum with respect to that index from 1 to $n$. It is noted that $\theta$ is the angle between a direction at a point $P$ on $S$ and the parametric curve $Y = \text{const.}$, through $P$ on $S$. Representing $dw/dZ$ in a plane, called the derivative plane $\Delta$, $dw/dZ$ is depicted as a clock with center vector $H + iK = \frac{\partial w}{\partial z^m} d\bar{z}$, and phase vector $h + ik = \frac{\partial w}{\partial z^m} d\bar{z}$.

This clock depends on the point $P$, the conformal surface $S$ through $P$, and the parameter $Z$ describing $S$.

If a change of the parameter $Z$ is performed, the central and phase vectors of the clock $r$ are multiplied by the same real number $p > 0$ and rotated through equal angles but in opposite directions. Denote this operation on a clock $r$ by $S^*(r)$.

Then all the clocks corresponding to a fixed point $P$ and a fixed surface $S$ through $P$ are the $2n$ clocks $S^*(r)$.

The totality of clocks at a given point $P$ is $2n$ but essentially there are $2n - 2$, one to each conformal surface $S$ through $P$.

If $\Gamma_1, \ldots, \Gamma_n$ are $n$ clocks belonging to $n$ distinct conformal surfaces $S_1, \ldots, S_n$, not all contained in the same $\Sigma_{2n-2}$, then any clock $\Gamma$ at $P$ is given by

$$\Gamma = S_1^*(\Gamma_1) + \ldots + S_n^*(\Gamma_n).$$

Thus the totality of clocks at a given point $P$ forms an abstract vector space of $n$ dimensions.

A set of $m$ clocks $\Gamma_1, \ldots, \Gamma_m$, at a given point $P$, is found to be linearly dependent if and only if they are obtained from $m$ conformal surfaces $S_1, \ldots, S_m$, through $P$, all of which belong to the same pseudo-conformal manifold $\Sigma_{2m-2}$.

A polygenic function $w$ is monogenic at a given point if and only if $dw/dZ$ is unique along each of $n$ distinct conformal surfaces through $P$, not all contained in the same $\Sigma_{2n-2}$.

This result generalizes the usual definition of a monogenic function.

A polygenic function $w$ is multiharmonic in $\Sigma_n$ if and only if it is multiharmonic over every $\Sigma_m$ where $m$ is fixed and $m \leq n$.

Also it may be shown that the center transformation is direct conformal for every $S$ if and only if $w$ is multiharmonic in $\Sigma_n$.

6. The pseudo-angle between a $(2n - 1)$ dimensional manifold $S_{2n-1}$: $F(x^1, \ldots, x^n; y^1, \ldots, y^n) = 0$, and a curve $Cx^\alpha$: $x^\alpha(t)$, where $\alpha = 1, \ldots, n$, at a given point $P$ of intersection, is

$$\theta = \arctan \frac{\frac{\partial F}{\partial x^\alpha} dx^\alpha + \frac{\partial F}{\partial y^\alpha} dy^\alpha}{\frac{\partial F}{\partial y^\alpha} dx^\alpha - \frac{\partial F}{\partial x^\alpha} dy^\alpha}.$$
The pseudo-angle characterizes the pseudo-conformal group $G$.\footnote{The term polygenic was introduced by Kasner in 1927. See: "A New Theory of Polygenic (or non-Monogenic) functions," Science, 66, 581–582 (1927). The term non-analytic is also used.}

Let $S_{2m}$ be a fixed $2m$ dimensional manifold contained in $\Sigma_{2n}$ so that $m \leq n$. Let $P$ be an arbitrary point of $S_{2m}$. Suppose that $S_{2n-1}$ is an arbitrary $(2n - 1)$ dimensional manifold in $\Sigma_{2n}$, which contains $P$, and let the intersection of $S_{2m}$ and $S_{2n-1}$ be an $S_{2m-1}$. Thus $S_{2m}$ is not contained in $S_{2n-1}$.

The pseudo-angle between any curve $C$, in $S_{2m}$, and $S_{2n-1}$ at $P$, is equal to that between $C$ and $S_{2m-1}$, for every $S_{2n-1}$, if and only if $S_{2m}$ is pseudo-conformal.

This gives a geometric characterization of the pseudo-conformal manifolds $\Sigma_{2m}$ contained in a given pseudo-conformal manifold $\Sigma_{2n}$, where $m \leq n$.

If $w = \phi + i\psi$ is monogenic over $R$, then the curves of $\phi = \text{const.}$ are pseudo-orthogonal to the manifolds $\psi = \text{const.}$

For $n = 1$, this reduces to the well-known result that the components of a monogenic function of a complex variable give rise to an orthogonal isothermal net.

\begin{itemize}
  \item \footnote{Poincaré, Compt. rend., 96, 238, (1883); Acta Math., 2, 99, (1883), 22, 112 (1898); Palermo Rendiconti (1907).}
  \item \footnote{Kasner, "Conformality in Connection with Functions of Two Complex Variables," Trans. Am. Math. Soc., 48, 50–62 (1940).}
  \item \footnote{Kasner and De Cicco, "The Geometry of Polygenic Functions," Rev. Math. Univ Tucuman (Argentina), 4, 7–45 (1944).}
\end{itemize}

THE ELEMENT OF VOLUME OF THE ROTATION GROUP

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The $n$-dimensional rotation group is an $\frac{n(n - 1)}{2}$-parameter group and if we set $n = 2k$ or $n = 2k + 1$, according as $n$ is even or odd, it is usually convenient to adopt as $k$ of these parameters the $k$ angles $\theta_1, \theta_2, \ldots, \theta_k$ which determine the class of the group to which the particular element $X$ of the group which we wish to specify belongs (the function of the remain-