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All of the crosses reported in this paper were between clones derived from single ascospores. We are indebted to Dr. C. C. Lindegren for the parent clones. The symbols refer to genes affecting the following characters: $a$ and $a$, mating type; $G$ and $g$, galactose fermentation and non-fermentation (within 96 hours), respectively; $F$ and $f$, flaky and free dispersion, respectively, in liquid medium. The flaky character was particularly clear-cut in DIFCO Yeast Nitrogen Base medium with 1% glucose added.


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**ON THE FUNCTIONAL EQUATIONS OF THE DIRICHLET SERIES DERIVED FROM SIEGEL MODULAR FORMS**

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The striking analogy between the formulas

$$
\int_0^\infty e^{-tx}e^{-1}dx = \Gamma(s)t^{-s}, \quad t > 0, \quad Re(s) > 0,
$$

(1a)

and

$$
\int_{X > 0} e^{-i\pi TX} |X|^{-\frac{n+1}{2}} dX = \pi^{\frac{n(n-1)}{2}} \Gamma(s)\Gamma\left(s - \frac{1}{2}\right) \ldots \times \\
\Gamma\left(s - \frac{n-1}{2}\right)|T|^{-s}, \quad Re(s) > \frac{n-1}{2},
$$

(1b)

where, in the second formula, the integration is over the space of positive definite $n \times n$ matrices, $|X|$ represents the determinant of $X$, and $T$ is a positive definite matrix, leads one to conjecture that there should be a theory connecting Siegel modular forms and Dirichlet series possessing certain types of functional equations, similar to that created by Hecke connecting Dirichlet series and modular forms of one variable.

The first step in the construction of such a theory has been taken by Maass, who showed that from modular forms of the second degree one can derive an infinite set of Dirichlet series possessing functional equations. Contrary to what occurs in the Hecke theory, the derivation of the functional equation is no longer a simple preliminary step, but requires some detailed analysis. Since the proof given by Maass is rather complicated,
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and applicable only in the case of modular forms of the second degree, we propose to present here a sketch of a simpler proof for the result of Maass, which possesses the merit of being able to be extended to the general case. Complete results will be presented subsequently.

Let \( F(Z) \) be a modular form of the second degree, satisfying the equation

\[
F((AZ + B)(CZ + D)^{-1}) = \left| CZ + D \right|^s F(Z)
\]

(2)

for \((AZ + B)(CZ + D)^{-1}\) an element of the modular group of the second degree, and possessing the Fourier expansion,

\[
F(X) = \sum_T a(T) e^{2\pi i \text{Tr}(XT)},
\]

(3)

where the summation is over all non-negative semi-integers, and the imaginary part of \( X \) is positive definite. If we use the parametric representation,

\[
\left( \frac{u(x^2 + y^2)}{y} \quad \frac{ux}{y} \quad \frac{uy}{y} \right), \quad u > 0, \quad y > 0, \quad -\infty < x < \infty,
\]

(4)

for a positive definite \( 2 \times 2 \) matrix, cf. Maass, two domains of importance below are

\[
D: \quad 0 < u \leq 1, \quad -\frac{1}{2} \leq x \leq \frac{1}{2}, \quad x^2 + y^2 \geq 1,
\]

\[
\Delta: \quad -\frac{1}{2} \leq x \leq \frac{1}{2}, \quad x^2 + y^2 \geq 1.
\]

Let us decompose \( F(X) \) into the sum of the two functions defined by

\[
F_1(X) = \sum_{|T|=0} a(T) e^{2\pi i \text{Tr}(TX)},
\]

(6)

where the summation is over all non-negative semi-integers of rank 0 or 1, and

\[
F_2(X) = \sum_{T>0} a(T) e^{2\pi i \text{Tr}(TX)},
\]

(7)

where the summation is over all positive semi-integers. Then, cf. Maass,

\[
\int_D \left| X \right|^{s-1} F_2(iX) dX = \int_{X>0} \left| X \right|^{s-1} \left( \sum_{|T|=0} a(T) e^{2\pi i \text{Tr}(TX)} \right) dX,
\]

(8)

where \( e(T) \) is the number of elements of the class of \( T \), and the summation on the right is over all distinct classes of \( 2 \times 2 \) positive-definite semi-integers. Using the case \( n = 2 \) of (1b), we obtain for \( \text{Re}(s) \) sufficiently large,

\[
f(s) = \pi^{1/2} \Gamma(s + 1/2) \sum_{|T|=0} a(T) \left| T \right|^{-(s+1/2)} = \int_D \left| X \right|^{s-1} \left[ F(iX) - F_1(iX) \right] dX
\]

(9)

\[
= \int_0^1 u^{s-1} (\int_{\Delta} [F(iX) - F_1(iX)]) du, \quad (\left| X \right| = u).
\]
At first glance, one is tempted to try the classical method of splitting the right-hand integral into two parts, one over $0 < X < 1$, and the other over $1 < X$, followed by a change of variable $X = X^{-1}$, and an application of the modular relation of (2). The presence of the non-constant term $F_i(iX)$ effectively stymies this approach.

To circumvent this difficulty, we use an alternative method based upon analytic continuation. Let us illustrate this method in its starkest simplicity by deriving the functional equation for the Riemann zeta-function. We have, for $\Re(s) > 1$,

$$\Gamma(s) \xi(2s) = \int_0^\infty \left( \sum_{n=1}^{\infty} e^{-n^2x} \right) x^{s-1} \, dx.$$  \hspace{1cm} (10)

Inverting by means of the Mellin inverse, shifting the line of integration to the left past the two poles at $s = 0$, and $s = 1/2$, taking account of the residues at these points, and then re-inverting, we obtain for $\Re(s) < 0$,

$$\Gamma(s) \xi(2s) = \int_0^\infty \left( \sum_{n=1}^{\infty} e^{-n^2x} + \frac{1}{2} - \frac{\sqrt{\pi}}{2} x^{-1/2} \right) x^{s-1} \, dx.$$  \hspace{1cm} (11)

Now applying the functional equation for the theta function, we obtain the functional equation for $\Gamma(s) \xi(2s)$. This method assumes that we have already demonstrated the analytic continuability of $\Gamma(s) \xi(2s)$ and knows its meromorphic character. All this can be obtained by relatively crude methods based upon the use of the modular property of the theta function. Similarly, below, we shall assume that the analytic properties we require have already been demonstrated, namely the analytic continuability, and the existence of a finite number of poles.

From (9), we obtain by use of the Mellin inversion formula,

$$\int_{\Delta} [F(iX) - F_i(iX)] = \frac{1}{2\pi i} \int_{C} f(s) u^{-s} \, ds,$$  \hspace{1cm} (12)

where $C$ is a straight line, $\Re(s) = a > s_0$. Shifting the contour to the left, past all the poles of $f(s)$, finite in number, we have for $\Re(s) < s_1 < 0$,

$$\int_{\Delta} [F(iX) - F_i(iX)] = R(u) + \frac{1}{2\pi i} \int_{C_1} f(s) u^{-s} \, ds,$$  \hspace{1cm} (13)

where $R(u)$ is a function of the type $\sum_{k=1}^{N} a_k u^{-b_k}$, and the integral is $0(u^0)$ as $u \to 0$, $u > 0$, for any $n > -s_1$.

On the other hand, we have, from (2),

$$\int_{\Delta} [F(iX) - F_i(iX)] = \int_{\Delta} [u^{-k} F_i(iX^{-1}) - F_i(iX)]$$

$$= u^{-k} \int_{\Delta} F_i(iX^{-1}) - \int_{\Delta} F_i(iX) + u^{-k} \int_{\Delta} F_i(iX^{-1})$$  \hspace{1cm} (14)
Using the inequality $tr(TX) \geq u \sqrt{t_0 t_2}$ for $(x, y) \in \Delta$, where
\[
T = \begin{pmatrix} t_0 & t_1 \\ t_1 & t_2 \end{pmatrix} > 0,
\]
(15)

it is easy to demonstrate that the third integral in (14) is $0(e^{-c/u})$ for some $c > 0$ as $u \to 0$, $u > 0$. If we can show that the first two integrals are of the forms $\sum_{k=1}^{n} c_k u^{-d_k}$, it will follow that the sum of the first two integrals must equal $R(u)$. Separating out the constant terms corresponding to $T = 0$, and using the parametric representation of $2 \times 2$ semidefinite matrices, we find, using the techniques contained in equations (25) and (78) of Maass’ paper that the integrals have the required form. Thus,
\[
\int_{\Delta} \left( \sum_{T>0} a(T)e^{-2\pi ir(TX)} \right) - R(u) = u^{-k} \int_{\Delta} F_2(iX^{-1})
\]
(16)

Hence, using (13), for $Re(s) < s_1$, we have
\[
f(s) = \int_0^1 u^{s-1-k} (\int_{\Delta} F_2(iX^{-1})) du
\]
\[
= \int X > 0 \left[ \sum_{[T]} \frac{a(T)}{\epsilon(T)} e^{-i\pi(TX^{-1})} \right] |X|^{s-k-1}
\]
(17)

The change of variable $X = X_1^{-1}$ and integration term-by-term yields the desired functional equation.

The general series discussed by Maass, involving the automorphic solutions of the equation
\[
\left( y^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{4} + r^2 \right) u = 0
\]
(18)

may be handled similarly.