\[
\log \frac{N(\Delta, r)}{mn} > e_0 bn \\
> l_{r-1}(ebn^2) \\
> l_r(mn)
\] (18)

Coupling (18) with (9), completes the proof of Theorem 1.

* Under the auspices of the Office of Naval Research.

1 Minkowski, H., Diophantische Approximationen, Leipzig, 1907.
5 Weyl, H., Algebraic Theory of Numbers, 1940, Princeton Univ. Press.

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**TYPOGRAPHICAL CORRECTION**

In my paper "On the Basis Theorem for Finite Abelian Groups (Third Note)," these PROCEEDINGS, 37, 611–614 (September, 1951), the symbols \( \varepsilon \) and \( \bar{\varepsilon} \) surmounted by a bar, \( \bar{\varepsilon} \), were used to denote, respectively, membership and non-membership of an element in a group. Owing to a printer's error, the bar is omitted over some of the \( \varepsilon \)'s, thus making it difficult to follow the logic of the proof. Bars should appear over the following \( \varepsilon \)'s:

- P. 611, line after "THEOREM," second \( \varepsilon \).
- P. 612, line 5 from bottom.
- P. 613, line 3 from top.

Also, on p. 611, the bar over the \( \varepsilon \) on the line following the lemma should be made more clear.

JESSE DOUGLAS
ON THE BASIS THEOREM FOR FINITE ABELIAN GROUPS
(THIRD NOTE)*

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Communicated July 6, 1951

1. Let $G$ denote any finite abelian group; we wish to prove the existence of a basis $(\theta_1, \theta_2, \ldots, \theta_m)$ for $G$. This is to require (in additive notation) that: (i) every element $\theta$ of $G$ shall be representable in the form

$$\theta = a_1\theta_1 + a_2\theta_2 + \ldots + a_m\theta_m \quad (1)$$

while (ii) (independence condition)

$$c_1\theta_1 + c_2\theta_2 + \ldots + c_m\theta_m = 0 \quad (2)$$

shall imply

$$c_i\theta_i = 0 \quad \text{for } i = 1, 2, \ldots, m. \quad (3)$$

0 denotes the identical element; small English letters, here and in the sequel, have integer values.

Because of the standard representation of any finite abelian group as direct product of subgroups of prime-power order, the case of prime-power order leads immediately to the general case; one has only to combine the bases of the component subgroups. This justifies us in supposing from now on that the order of $G$ is $p^\alpha$, $p$ a prime, $\alpha$ a positive integer.

By a $B$-group we shall understand a subgroup of $G$ that has a basis. Certainly $B$-groups exist, e.g., the cyclic subgroups of $G$.

Our goal is to prove the following

**Theorem.** $G$ itself is a $B$-group.

We shall use $\epsilon$ to denote membership of an element in a group, $\epsilon$ non-membership. By $H < K$, or $K > H$, we shall mean that $H$ is a *proper* subgroup of $K$; $H$ contained in $K$, but $H \neq K$.

2. **Lemma.** If $H$ is a $B$-group and $H < G$, then a $B$-group $K$ exists such that $H < K$.

**Proof:** By hypothesis, an element $\varphi$ of $G$ exists such that $\varphi \in H$. But $p^\lambda \varphi \in \bar{H}$ if $\lambda$ is large enough, e.g., if $p^\lambda = \text{period of } \varphi$, so that $p^\lambda \varphi = 0$. Let $\lambda > 0$ be fixed so that $p^\lambda$ is the *lowest* power of $p$ such that $p^\lambda \varphi \in H$. Then

$$p^\lambda \varphi = a_1\theta_1 + a_2\theta_2 + \ldots + a_m\theta_m, \quad (4)$$

where $(\theta_1, \theta_2, \ldots, \theta_m)$ is a basis of $H$. We suppose this basis to be without zero elements, and written in a descending order (wide sense) of the respective periods:

$$p^{k_1} \geq p^{k_2} \geq \ldots \geq p^{k_m}.$$
Since this is equivalent to
\[
k_1 \geq k_2 \geq \ldots \geq k_m,
\]
each period is a multiple of all the following ones. A consequence is that \(d\theta_r = 0\) implies
\[
d\theta_{r+1} = 0, d\theta_{r+2} = 0, \ldots, d\theta_m = 0. \tag{5}
\]
In the proof of our lemma there are two cases.

3. Case I: Each coefficient \(a_i\) in (4) is divisible by \(p\): \(a_i = pb_i\) for \(i = 1, 2, \ldots, m\).

Then (4) can be written
\[
\phi(p^{\lambda-1} \varphi - b_1\theta_1 - b_2\theta_2 - \ldots - b_m\theta_m) = 0. \tag{6}
\]
Let
\[
\omega = p^{\lambda-1} \varphi - b_1\theta_1 - b_2\theta_2 - \ldots - b_m\theta_m; \tag{7}
\]
then, first, \(\omega \in H\); otherwise \(p^{\lambda-1} \varphi \in H\), contrary to our least power hypothesis concerning \(p^\lambda\).

Also
\[
\phi\omega = 0. \tag{8}
\]
From this we can infer that \(c\omega \in H\) implies \(c\omega = 0\). For if \(c\omega \neq 0\), \(p\) does not divide \(c\); hence \(xc + yp = 1\) for certain integers \(x, y\); therefore \(\omega = (xc + yp)\omega = x(c\omega) + y(p\omega) = x(c\omega)\); thus \(\omega \in H\), contrary to our known condition.

It can now be proved that
\[
(\omega, \theta_1, \theta_2, \ldots, \theta_m) \tag{9}
\]
are independent elements, and therefore form a basis of the group \(K\) which they generate.

For suppose
\[
c\omega + c_1\theta_1 + c_2\theta_2 + \ldots + c_m\theta_m = 0.
\]
Then \(c\omega \in H\), hence (by the second preceding paragraph) \(c\omega = 0\); consequently also \(c_1\theta_1 = 0, \ldots, c_m\theta_m = 0\), because of the independence of the \(\theta\)'s. Thus all the elements (9) are independent.

Accordingly, \(K\), possessed of a basis, is a \(B\)-group. But \(K\) contains a basis of \(H\), and therefore \(H\) itself—also the element \(\omega \in H\). Thus \(K > H\).

4. Case II: At least one coefficient \(a_i\) in (4) is not divisible by \(p\); let \(a_r\) be the first such coefficient.

We have then from (4)
\[
\phi(p^{\lambda-1} \varphi - b_1\theta_1 - \ldots - b_{r-1}\theta_{r-1}) = a_r\theta_r + \ldots + a_m\theta_m. \tag{10}
\]
Let
\[ \omega = p^{\lambda-1} \varphi - b_1 \theta_1 - \ldots - b_{r-1} \theta_{r-1}; \]  
(11)
then \( \omega \in H \), otherwise \( p^{\lambda-1} \varphi \in H \), contrary to hypothesis (cf. the text just preceding (4)).

Also, by (10),
\[ p\omega = a_r \theta_r + a_{r+1} \theta_{r+1} + \ldots + a_m \theta_m, \]
(12)
where \( p \) does not divide \( a_r \). Therefore integers \( x, y \) exist such that
\[ xa_r + yp^{kr} = 1. \]
(13)
Here \( p^{kr} \) is the period of \( \theta_r \), so that
\[ 0 = p^{kr} \theta_r. \]
(14)
By linear combination of (12) and (14) with the multipliers \( x, y \), we get
\[ xp\omega = \theta_r + xa_{r+1} \theta_{r+1} + \ldots + xa_m \theta_m. \]

It follows that the group \( K \) generated by
\[ (\theta_1, \theta_2, \ldots, \theta_{r-1}, \omega, \theta_{r+1}, \ldots, \theta_m) \]
(15)
includes \( \theta_r \), therefore includes all of \( H \).

But \( K \) also contains the element \( \omega \in H \); hence \( K > H \).

It remains to prove that \( K \) is a \( B \)-group. This we shall do by showing that the generators (15) of \( K \) are independent, and so form a basis of \( K \).

Indeed, suppose that
\[ c_1 \theta_1 + c_2 \theta_2 + \ldots + c_{r-1} \theta_{r-1} + c_\omega + c_{r+1} \theta_{r+1} + \ldots + c_m \theta_m = 0. \]
(16)
Then \( c_\omega \in H \); also \( p\omega \in H \), by (12). Hence \( p \) divides \( c \): \( c = dp \)—otherwise \( xc + yp = 1 \) for certain integers \( x, y \); \( \omega = (xc + yp)\omega = x(c\omega) + y(p\omega) \); therefore \( \omega \in H \), contrary to the statement following (11).

Consequently, with use of (12),
\[ c\omega = dp\omega = da_\theta \theta_r + da_{r+1} \theta_{r+1} + \ldots + da_m \theta_m. \]
(17)
Making this substitution in (16), we find a linear relation in the \( \theta \)'s wherein the term in \( \theta_r \) is \( da_\theta \theta_r \); by the independence of the \( \theta \)'s, it follows that
\[ da_\theta \theta_r = 0. \]
(18)
Since, by (14),
\[ dp^{kr} \theta_r = 0, \]
(19)
we obtain with use of (13): \( d\theta_r = 0 \); hence, by (5),
\[ d\theta_{r+1} = 0, \ldots, d\theta_m = 0. \]
(20)
By (17), then,
\[ c_\omega = 0. \]  
(21)

From the independence of the \( \theta \)'s, it follows that the other terms in (16) are individually = 0, thus completing our proof of independence of the elements (15).

5. Let now \( H_0 \) be a \( B \)-group of maximum order; certainly \( H_0 \) exists, since the orders of the \( B \)-groups are bounded by the finite order of \( G \).

We say \( H_0 = G \), thus proving our main theorem.

For if \( H_0 < G \), we are in obvious contradiction with our lemma.

* Recent treatments of this "fundamental theorem of abelian groups" are: by the author, these PROCEEDINGS, 37, 359-362, 525-528 (1951); by Rado, R., J. London Math. Soc., 26 (part I), 74-75 (1951). These new proofs are definitely superior in simplicity and directness to those previously given in the standard treatises on algebra and group theory.

1 This definition allows any number of zero elements to be adjoined to or removed from a basis. Thus we may always suppose a basis to contain no zero elements—except in the case of the group consisting solely of 0, whose basis we shall regard as 0.

That the elements of a basis (without zero elements) are necessarily distinct follows immediately from the independence condition.

2 No proof of our lemma is required in the case of the group \( H \) consisting solely of 0, for this is properly contained in every other cyclic subgroup of \( G \), i.e., in a \( B \)-group.

HARMONIC TENSORS ON MANIFOLDS WITH BOUNDARY

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Communicated by S. Lefschetz, July 28, 1951

1. Introduction.—The boundary-value problems considered in this note relate to harmonic \( p \)-tensors on a Riemannian manifold with boundary. We state certain theorems of existence and uniqueness, which are solutions for the boundary-value problems of Dirichlet and Neumann type for harmonic \( p \)-tensors on a boundary manifold. The results which we state, with the exception of Theorem 8, were conjectured by Tucker. Proofs of the theorems will appear in the Annals of Mathematics.

We consider alternating (skew-symmetric) covariant tensors \( \varphi_{i_1 \ldots i_p} \) of rank \( p \) on a Riemannian manifold \( M \) with boundary \( B \), and also their associated differential forms,

\[ \varphi = \sum_{i_1 < \ldots < i_p} \varphi_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}. \]