[3] See §9, 10 in a forthcoming paper on Track Groups by M. G. Barratt, concerning the homotopy groups of function spaces $X^P$. The group in question is

$$\pi_{n-2}\ [(S^{n-1})^{Y^2}, x_0]$$

where $x_0$ is an inessential map.

[4] Whitehead, J. H. C., "A Note on Suspension," *Q. J. Math. (Oxford)* (2), **1**, 9–22 (1950). See also reference 2.

[5] See reference 3, §2. A map $\phi: P \rightarrow Q$ induces a homomorphism $\phi^*: \pi_k(X^Q) \rightarrow \pi_k(X^P)$ for a large class of spaces $P, Q, X$.

[6] Hilton, P. J., "The Calculation of Homotopy Groups of $A_n^2$-polyhedra (II)," *Ibid.*, (2), **2**, 228–240 (1951). Theorem 1.3 settles the problems stated on pages 234 and 240.

[7] Eckmann, B., "Zur Homotopietheorie Gefaserter Räume," *Comm. Math. Helvetici*, **14**, 141–192 (1941); also "Uber Homotopiegruppen fon Gruppenräumen," *Ibid.*, **14**, 234–256 (1941).

[8] Whitehead, J. H. C., "On the Groups $\pi_r(V_{n, m})$ and Sphere Bundles," *Proc. London Math. Soc.*, **48**, 243–291 (1943).

[9] The key theorems of Eckmann and Whitehead are Theorem 12 in reference 8, and Theorems 7, 9 and the construction on pages 270–271 in reference 9.

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# *FIXED-POINT AND MINIMAX THEOREMS IN LOCALLY CONVEX TOPOLOGICAL LINEAR SPACES**

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1. Let $K$ be a compact convex set in a Euclidean space and $\mathfrak{K}(K)$ be the family of all closed convex (non-empty) subsets of $K$. Kakutani's fixed-point theorem[1] asserts that, for any upper semicontinuous point-to-set transformation $f$ from $K$ into $\mathfrak{K}(K)$, there exists a point $x_0 \in K$ such that $x_0 \in f(x_0)$. Here upper semicontinuity means that $\lim x_n = x_0$, $y_n \in f(x_n)$ and $\lim y_n = y_0$ imply $y_0 \in f(x_0)$. On the other hand, A. Tychonoff[2] has shown that, if $K$ is a compact convex set in a locally convex topological linear space $L$, then any continuous point-to-point transformation from $K$ into itself has at least one fixed point. In the present note, Theorem 1 is a common generalization of both theorems of Kakutani and Tychonoff. Like Tychonoff, we consider a compact convex set $K$ in a locally convex topological linear space $L$. Like Kakutani, we deal with upper semicontinuous point-to-set transformations from $K$ into $\mathfrak{K}(K)$. However, since the first countability axiom is not assumed for the space $L$, we have to define the upper semicontinuity in terms of open sets, not in terms of convergent sequences. (The two definitions are equivalent for metric spaces.) Recently H. F. Bohnenblust and S. Karlin[3] have extended Kakutani's theorem to Banach spaces along the same lines as Schauder's fixed-point theorem[4] generalizes Brouwer's; their result does not include Tychonoff's theorem.

Theorems 2 and 3 of this note are applications of Theorem 1. In the case of two Euclidean spaces, Theorem 2 was first proved by von Neumann.[5] Theorem 3 extends von Neumann's minimax theorem[6] to locally convex topological linear spaces, and thus includes also the minimax theorem of J. Ville.[7]

2. As a topological linear space is not always metrizable, but is always a separated uniform space,[8] we shall need the following

LEMMA 1. *Let $X$ be a separated uniform space. Let $A$ be a closed set and $B$ a compact set in $X$. For any entourage $U$ of the uniform structure of $X$, there exists another entourage $V$ such that*

$$V(A) \cap V(B) \subset U(A \cap B), \tag{1}$$

*where $V(A) = \{x \in X | (a, x) \in V \text{ for some } a \in A\}$ and $V(B)$, $U(A \cap B)$ have similar meaning.*

*Proof:* Choose an open entourage $U_1$, such that $U_1^2 \subset U$. Then $U_1(A \cap B)$ is open. Let $C$ be the set of all points of $B$ not belonging to $U_1(A \cap B)$. Then $C$ is a compact set disjoint from the closed set $A$. We can find an entourage $U_2$ such that $C \cap U_2(A) = \phi$, which implies $B \cap U_2(A) \subset U_1(A \cap B)$. If $V$ is an entourage such that $V = V^{-1} \subset U_1$ and $V^2 \subset U_2$, then (1) is satisfied. In fact, for any $x \in V(A) \cap V(B)$, we have $(a, x) \in V$, $(b, x) \in V$ with $a \in A$, $b \in B$. Since $V = V^{-1}$, we have $(a, b) \in V^2 \subset U_2$ and therefore $b \in B \cap U_2(A) \subset U_1(A \cap B)$. There exists $c \in A \cap B$ such that $(c, b) \in U_1$. This together with $(b, x) \in V$ implies $(c, x) \in V \cdot U_1 \subset U_1^2 \subset U$ or $x \in U(c) \subset U(A \cap B)$.

Let $X$, $Y$ be two topological spaces and let $\mathfrak{C}(Y)$ denote the family of all closed sets in $Y$. A point-to-set transformation $f$ from $X$ into $\mathfrak{C}(Y)$ is called *upper semicontinuous* (u. s. c.), if, for any point $x_0 \in X$ and any open set $U$ in $Y$ such that $f(x_0) \subset U$, there exists a neighborhood $W$ of $x_0$ such that $f(x) \subset U$ for all $x \in W$.

THEOREM 1. *Let $L$ be a locally convex topological linear space[9] and $K$ a compact convex set in $L$. Let $\mathfrak{K}(K)$ be the family of all closed convex (non-empty) subsets of $K$. Then for any u. s. c. point-to-set transformation $f$ from $K$ into $\mathfrak{K}(K)$, there exists a point $x_0 \in K$ such that $x_0 \in f(x_0)$.*

*Proof:* Let $\mathfrak{B}$ be a fundamental system of neighborhoods of the null-element 0 in $L$. $L$ being locally convex, we may assume that each $V \in \mathfrak{B}$ is convex, open and such that $V = -V$. For each $V \in \mathfrak{B}$, let

$$F_V = \{x \in K | x \in f(x) + \bar{V}\}.$$

If we can prove that each $F_V$ is non-empty and closed, then it will follow that also any finite intersection of them is non-empty. The compactness

of $K$ will then imply $\bigcap_{V \epsilon \mathfrak{V}} F_V \neq \phi$. Clearly any point $x_0$ in this intersection will satisfy $x_0 \epsilon f(x_0)$.

Consider an arbitrarily fixed $V \epsilon \mathfrak{V}$. As $K$ is compact, there exists a finite number of points $z_1, z_2, \ldots, z_n$ of $K$ such that $K \subset \bigcup_{i=1}^{n} (V + z_i)$. Let $C$ be the smallest closed convex set containing $z_1, z_2, \ldots, z_n$. Let $\mathfrak{K}(C)$ be the family of all closed convex (non-empty) subsets of $C$. For each $x \epsilon C$, we define

$$f_V(x) = (f(x) + \bar{V}) \cap C.$$

Since $V$ is convex, it can be easily seen that $f_V$ is a transformation from $C$ into $\mathfrak{K}(C)$. We claim that $f_V$ is u. s. c. Let $x_0 \epsilon C$ and let $U$ be an open set in $L$ such that $f_V(x_0) \subset U$. As $f_V(x_0)$ is compact, we can find a $V_1 \epsilon \mathfrak{V}$ such that $V_1 + f_V(x_0) \subset U$. By Lemma 1, there exists a $V_2 \epsilon \mathfrak{V}$ such that

$$(V_2 + f(x_0) + \bar{V}) \cap (V_2 + C) \subset V_1 + [(f(x_0) + \bar{V}) \cap C].$$

We have then

$$(V_2 + f(x_0) + \bar{V}) \cap C \subset V_1 + f_V(x_0) \subset U.$$

Now, $f$ being u. s. c., there is a neighborhood $W$ of $x_0$ such that $f(x) \subset V_2 + f(x_0)$ for all $x \epsilon W \cap K$. Hence, for $x \epsilon W \cap C$, we have

$$f_V(x) = (f(x) + \bar{V}) \cap C \subset (V_2 + f(x_0) + \bar{V}) \cap C \subset U.$$

This proves that $f_V$ is u. s. c. By Kakutani's theorem, there is a point $x_0 \epsilon C$ such that $x_0 \epsilon f_V(x_0) \subset f(x_0) + \bar{V}$. Hence $F_V \neq \phi$.

Let $y \epsilon K$ be not contained in $F_V$. Then $y$ is not contained in the closed set $f(y) + \bar{V}$. There is a $V_3 \epsilon \mathfrak{V}$ such that

$$(y + V_3) \cap (f(y) + \bar{V} + V_3) = \phi.$$

Again since $f$ is u. s. c., there is a $V_4 \epsilon \mathfrak{V}$ such that $f(z) \subset f(y) + V_3$ for all $z \epsilon (y + V_4) \cap K$. We may assume $V_4 \subset V_3$. Then for any $z \epsilon (y + V_4) \cap K$, $z$ is not in $f(z) + \bar{V}$, nor in $F_V$. This shows that $F_V$ is closed and completes the proof.

3. LEMMA 2. *Let $X$, $Y$ be two topological spaces, of which $Y$ is compact. Let $E$ be a closed set in the product space $X \times Y$. For any $x \epsilon X$, let $E(x)$ be the section of $E$ determined by $x$: $E(x) = \{y \epsilon Y | (x, y) \epsilon E\}$. Then $x \to E(x)$ is an u. s. c. transformation from $X$ into $\mathfrak{C}(Y)$.*

*Proof:* For any subset $A$ of $X$, we use the notation $E(A) = \bigcup_{a \epsilon A} E(a)$. Let $\mathfrak{W}$ be the family of all neighborhoods of a point $x_0 \epsilon X$. Then we have $E(x_0) = \bigcap_{W \epsilon \mathfrak{W}} \overline{E(W)}$. Now, if $U$ is an open set in $Y$ containing $E(x_0)$, and if $F$ denotes the complement of $U$ in $Y$, then $F \cap \left[\bigcap_{W \epsilon \mathfrak{W}} \overline{E(W)}\right] = \phi$, and,

since $Y$ is compact, there exists a finite number of neighborhoods $W_1$, $W_2, \ldots, W_n$ of $x_0$ such that $F \cap \left[\bigcap_{i=1}^{n} \overline{E(W_i)}\right] = \phi$. Thus $W = \bigcap_{i=1}^{n} W_i$ is a neighborhood of $x_0$ such that $E(W) \subset U$.

LEMMA 3. *Let $X$ be a topological space and $\{Y_\nu\}_{\nu\epsilon I}$ be a family of compact spaces. Let $Y = \prod_{\nu\epsilon I} Y_\nu$ be the product space. If, for each $\nu \epsilon I$, $f_\nu$ is an u. s. c. transformation from $X$ into $\mathfrak{C}(Y_\nu)$, then the transformation $f$ from $X$ into $\mathfrak{C}(Y)$ defined by $f(x) = \prod_{\nu\epsilon I} f_\nu(x)$ is also u. s. c.*

*Proof:* Consider first the case of a finite set $I = \{1, 2, \ldots, n\}$. Let $x_0 \epsilon X$ and let $U$ be an open set in $Y$ such that $f(x_0) = \prod_{i=1}^{n} f_i(x_0) \subset U$. By a theorem of A. Weil, the compact space $Y_i$ can be regarded as a uniform space.[8] Since $f_i(x_0)$ is a compact set in the uniform space $Y_i$, and $U$ is an open set in $Y$ containing $\prod_{i=1}^{n} f_i(x_0)$, there exists for each $i$ an open set $U_i$ in $Y_i$ such that $f_i(x_0) \subset U_i$ $(1 \leqq i \leqq n)$ and $\prod_{i=1}^{n} U_i \subset U$. As $f_i$ is u. s. c., there is a neighborhood $W_i$ of $x_0$ such that $f_i(x) \subset U_i$ for $x \epsilon W_i$. Then $W = \bigcap_{i=1}^{n} W_i$ is a neighborhood of $x_0$ such that $f(x) = \prod_{i=1}^{n} f_i(x) \subset \prod_{i=1}^{n} U_i \subset U$ for all $x \epsilon W$.

Next consider the case of an infinite set $I$. Let $x_0 \epsilon X$ and let $U$ be an open set in $Y$ such that $f(x_0) = \prod_{\nu\epsilon I} f_\nu(x_0) \subset U$.

If $f_{\nu_0}(x_0) = \phi$ for some $\nu_0 \epsilon I$, then, as $f_{\nu_0}$ is u. s. c., we can find a neighborhood $W$ of $x_0$ such that $f_{\nu_0}(x) = \phi$ for all $x \epsilon W$. This implies that $f(x) = \prod_{\nu\epsilon I} f_\nu(x) = \phi \subset U$ for all $x \epsilon W$.

Assume now $f_\nu(x_0) \neq \phi$ for each $\nu \epsilon I$. Since $U$ is a union of elementary open sets (i.e., basic prisms) of $Y$ and $\prod_{\nu\epsilon I} f_\nu(x_0)$ is compact, there exists a finite number of elementary open sets $U^{(j)}$ in $Y$ $(1 \leqq j \leqq m)$ such that

$$\prod_{\nu\epsilon I} f_\nu(x_0) \subset \bigcup_{j=1}^{m} U^{(j)} \subset U. \tag{2}$$

These $m$ sets $U^{(j)}$ are of the form

$$U^{(j)} = \prod_{i=1}^{n} U_{\nu_i}^{(j)} \times \prod_{\substack{\nu\epsilon I \\ \nu \neq \nu_i}} Y_\nu,$$

where $U_{\nu_i}^{(j)}$ is open in $Y_{\nu_i}$. As each $f_\nu(x_0) \neq \phi$, (2) implies

$$\prod_{i=1}^{n} f_{\nu_i}(x_0) \subset \bigcup_{j=1}^{m} \prod_{i=1}^{n} U_{\nu_i}^{(j)},$$

where the right-hand side is an open set in $\prod_{i=1}^{n} Y_{\nu_i}$. From what we have just proved for the case of a finite set $I$, the transformation $x \to \prod_{i=1}^{n} f_{\nu_i}(x)$ from $X$ into $\mathfrak{C}\left(\prod_{i=1}^{n} Y_{\nu_i}\right)$ is u. s. c. Hence, there exists a neighborhood $W$ of $x_0$ such that

$$\prod_{i=1}^{n} f_{\nu_i}(x) \subset \bigcup_{j=1}^{m} \prod_{i=1}^{n} U_{\nu_i}^{(j)} \text{ for all } x \in W.$$

This implies that

$$f(x) = \prod_{\nu \in I} f_\nu(x) \subset \bigcup_{j=1}^{m} U^{(j)} \subset U \text{ for all } x \in W.$$

THEOREM 2. *Let $\{L_\nu\}_{\nu \in I}$ be a family (finite or infinite, not necessarily countable) of locally convex topological linear spaces. For each $\nu \in I$, let $K_\nu$ be a compact convex set in $L_\nu$ and let $H_\nu = \prod_{\substack{\lambda \in I \\ \lambda \neq \nu}} K_\lambda$. Let $K = \prod_{\nu \in I} K_\nu$ and let $\{E_\nu\}_{\nu \in I}$ be a family of closed subsets of $K$. If, for any point $x \in K$ and for any $\nu \in I$, the set in $K_\nu$*

$$pr_{K_\nu}[E_\nu \cap (K_\nu \times pr_{H_\nu} x)] \tag{3}$$

*is non-empty and convex, then $\bigcap_{\nu \in I} E_\nu \neq \phi$.*

*Proof:* As $K = K_\nu \times H_\nu$, the set (3) is the section of $E_\nu$ determined by the point $pr_{H_\nu} x$ of $H_\nu$, and therefore may be denoted by $E_\nu(pr_{H_\nu} x)$. By Lemma 2, the transformation $pr_{H_\nu} x \to E_\nu(pr_{H_\nu} x)$ from $H_\nu$ into $\mathfrak{C}(K_\nu)$ is u. s. c. On the other hand, the projection from $K$ onto $H_\nu$ is a continuous point-to-point transformation. Hence the transformation $x \to E_\nu(pr_{H_\nu} x)$ from $K$ into $\mathfrak{C}(K_\nu)$ is u. s. c. and therefore, by Lemma 3, the transformation $f$ from $K$ into $\mathfrak{C}(K)$ defined by $f(x) = \prod_{\nu \in I} E_\nu(pr_{H_\nu} x)$ is u. s. c. Furthermore, as $E_\nu(pr_{H_\nu} x)$ is non-empty and convex, we have $f(x) \in \mathfrak{K}(K)$. By Theorem 1, there exists a point $x \in K$ such that $x \in f(x)$, which means $x \in \bigcap_{\nu \in I} E_\nu$.

Using the case $I = \{1, 2\}$ of Theorem 2, one can easily prove:

THEOREM 3. *Let $L_1$, $L_2$ be two locally convex topological linear spaces, and $K_1$, $K_2$ be two compact convex sets in $L_1$, $L_2$, respectively. Let $f$ be a real-valued continuous function on $K_1 \times K_2$. If, for any $x_0 \in K_1$, $y_0 \in K_2$, the sets $\{x \in K_1 | f(x, y_0) = \operatorname{Max}_{\xi \in K_1} f(\xi, y_0)\}$ and $\{y \in K_2 | f(x_0, y) = \operatorname{Min}_{\eta \in K_2} f(x_0, \eta)\}$ are convex, then*

$$\operatorname*{Max}_{x \in K_1} \operatorname*{Min}_{y \in K_2} f(x, y) = \operatorname*{Min}_{y \in K_2} \operatorname*{Max}_{x \in K_1} f(x, y).$$

Theorem 3 includes Ville's minimax theorem, which we state in a much more general form: *Let* $X_1$, $X_2$ *be two compact spaces and let* $\varphi$ *be a real-valued continuous function on* $X_1 \times X_2$. *Let* $M_i^*$ $(i = 1, 2)$ *be the set of all regular Borel measures on* $X_i$ *with total measure* 1. *Then*[10]

$$\operatorname*{Max}_{\mu_1 \in M_1^*} \operatorname*{Min}_{\mu_2 \in M_2^*} \int_{X_1 \times X_2} \varphi d(\mu_1 \times \mu_2) = \operatorname*{Min}_{\mu_2 \in M_2^*} \operatorname*{Max}_{\mu_1 \in M_1^*} \int_{X_1 \times X_2} \varphi d(\mu_1 \times \mu_2).$$

We also mention that a direct application of Theorem 2 yields the following result on systems of linear equations with dominant diagonal coefficients: *Let* $\sum_{\nu \in I} a_{\lambda\nu} x_\nu = b_\lambda (\lambda \in I)$ *be a system (finite or infinite, not necessarily countable) of linear equations with real coefficients such that for each fixed* $\lambda \in I$, *the family* $\{a_{\lambda\nu}\}_{\nu \in I}$ *is summable.*[11] *If* $a_{\lambda\lambda} \geqq \sum_{\substack{\nu \in I \\ \nu \neq \lambda}} |a_{\lambda\nu}| + |b_\lambda|$ *for each* $\lambda \in I$, *then the system has a solution* $\{x\}_{\nu \in I}$ *such that* $-1 \leqq x_\nu \leqq 1$ *for each* $\nu \in I$.

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[1] Kakutani, S., "A Generalization of Brouwer's Fixed Point Theorem," *Duke Math. J.*, **8**, 457–459 (1941).

[2] Tychonoff, A., "Ein Fixpunktsatz," *Math. Ann.*, **111**, 767–776 (1935).

[3] Bohnenblust, H. F., and Karlin, S., "On a Theorem of Ville," *Contributions to the Theory of Games*, edited by H. W. Kuhn and A. W. Tucker, Princeton University Press, Princeton, 1950, pp. 155–160.

[4] Schauder, J., "Der Fixpunktsatz in Funktionalräumen," *Studia Math.*, **2**, 171–180 (1930).

[5] Von Neumann, J., "Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes," *Ergebnisse eines Math. Kolloqu.*, **8**, 73–83 (1937).

[6] Von Neumann, J., "Zur Theorie der Gesellschaftsspiele," *Math. Ann.*, **100**, 295–320 (1928); von Neumann, J., and Morgenstern, O., *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, 1944, pp. 153–155.

[7] Ville, J., "Sur la Théorie générale des Jeux où intervient l'Habileté des Joueurs," *Traité du Calcul des Probabilités et de ses Applications*, IV, 2; by E. Borel and collaborators, Gauthier-Villars, Paris, 1938, pp. 105–113.

[8] For topological terms and notations, we agree with Bourbaki, N., *Topologie Générale*, Chap. I–II, Hermann, Paris, 1940. Especially, for the theory of uniform spaces, see Chap. II of this book.

[9] A *topological linear space* is a real vector space topologized with a separated topology such that the vector operations are continuous. A topological linear space $L$ is *locally convex*, if the null-element of $L$ has a fundamental system (not necessarily countable) of neighborhoods formed exclusively by convex sets.

[10] Let $C_i^*$ $(i = 1, 2)$ be the conjugate space of the Banach space $C_i$ of all real-valued continuous functions on $X_i$. Then, with respect to the $w^*$-topology (i.e., the weak topology of $C_i^*$ induced from $C_i$), $C_i^*$ is a locally convex topological linear space, in which $M_i^*$ is a compact convex set. Moreover $\int_{X_1 \times X_2} \varphi d\ (\mu_1 \times \mu_2)$ is a continuous function on $M_1^* \times M_2^*$ (with respect to the $w^*$-topology).

[11] A family $\{a_\nu\}_{\nu \in I}$ of real numbers is *summable* and has *sum* $s$, if, for any $\epsilon > 0$, there exists a finite subset $J_0$ of $I$ such that $|s - \sum_{\nu \in J} a_\nu| < \epsilon$ for every finite subset $J \supset J_0$ of $I$. The sum $s$ is denoted by $\sum_{\nu \in I} a_\nu$.