where $b$ is 1, 3, or an odd divisor of $p^e - 1$ and $c$ and $e$ are rational integers, $c \geq 0$ and $e < \phi(b)$.

By hypothesis $(\rho) = \mathfrak{p}$, a prime ideal of $E$. But $\alpha = a\varepsilon$ since $p = \mathfrak{p}$, so that

$$\alpha = a \quad \text{or} \quad \alpha = a(1 - \rho),$$

where $a$ is a rational integer. Let $a = 2\varepsilon b$, $b$ odd. Since $b^\varepsilon \mathfrak{p}$, $F_o \supseteq F_b \supseteq R(\rho)$, if $b > 1$.

Hence by 4° and Theorem 1, $C_\mathfrak{p}(x)$ is reducible in $E/\mathfrak{p}$; and if $b \neq 3$, $p^e \equiv 1 \pmod{b}$, where $e$ is the common degree of the irreducible factors of $C_\mathfrak{p}(x)$ in $E/\mathfrak{p}$.

If $p \equiv 1 \pmod{3}$, then $p = N\rho$, $\mathfrak{p}$ a prime ideal of $E$. In this case the apparition problem is an open question.

3 For the three-term sigma formula, and all of the formulas of the previous paragraph, cf. Tannery and Molk, Éléments de la théorie des fonctions elliptiques, Vol. 2, pp. 234–236.

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**SOME THEOREMS ON PIECEWISE LINEAR EMBEDDING**

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1. We call two (Euclidean) polyhedra equivalent if they have isomorphic simplicial subdivisions; the homeomorphism which maps each simplex of such a subdivision linearly onto its correlate in an isomorphic subdivision is called a piecewise linear homeomorphism onto, abbreviated PLO.

A polyhedron is called finite if it has a subdivision consisting of a finite number of simplices; a polyhedron equivalent to a $q$-simplex is called a $q$-element, one equivalent to the boundary of a $q$-simplex, a $(q - 1)$-sphere. A polyhedron for which the star of every vertex of a given subdivision is a $q$-element is called a $q$-manifold: but in what follows "$q$-manifold" will mean "connected $q$-manifold." If $M^q$ is an orientable $q$-manifold, we denote by $\mathbf{M}^q$ the oriented manifold obtained by orienting
$M^q$ in one of the two possible ways; if $M^q$, $N^q$ are both oriented $q$-manifolds and if $\phi: M^q \to N^q$ is a PLO we shall write $\phi: M^q \to N^q$ if $\phi$ and the orientation of $M^q$ induce in $N^q$ the orientation of $N^q$. If $\phi: M^q \to M^q$ is an orientation preserving PLO we call it a $+PLO$.

In the following sections all theorems will be stated without proof; a full account will be published elsewhere.

2. By $R^n$ we denote Euclidean $n$-space. The set of $+PLOs \phi: R^n \to R^n$ form a group of transformations which we denote by $G^n$; by $G^n_0$ we denote the (normal) subgroup of $G^n$ consisting of those $\phi \in G^n$ which leave points outside some finite region of $R^n$ invariant (i.e., there must be some such region for each $\phi \in G^n$).

If $P$, $Q$ are (oriented or unoriented) polyhedra of $R^n$ we say that $P$, $Q$ are congruent in $R^n$, $P \equiv Q$ in $R^n$, if there is $\phi \in G^n$ such that $\phi P = Q$.

**Theorem A.** Let $P$, $Q$ be finite $q$-dimensional polyhedra of $R^n$, let $\psi$: $P \to Q$ be a given PLO and let $2q + 2 \leq n$. Then there is $\phi \in G^n$ such that $\phi | P = \psi$.

Thus, in particular, equivalent $q$-dimensional finite polyhedra of $R^n$ are congruent in $R^n$ if $2q + 2 \leq n$.

**Theorem B.** Let $P$, $Q$ be finite polyhedra of $R^n$, and let $\phi \in G^n$ be such that $\phi P = Q$. Then there is $\phi_0 \in G^n_0$ such that $\phi_0 | P = \phi | P$.

3. We say that a $q$-element (a $q$-sphere) of $R^n$ is flat in $R^n$ if it is congruent in $R^n$ to a $q$-simplex (the boundary of a $(q + 1)$-simplex).

**Theorem C.** Let $E^n$ be a $q$-element of $R^n$ and $K$ a given simplicial subdivision of $E^n$. Then $E^n$ is flat in $R^n$ if and only if the star of every vertex of $K$ is flat in $R^n$.

For $q < n$, we denote by $[g, n]$ the set of congruence classes of oriented $q$-elements of $R^n$; for $q < n - 1$ we denote by $(g, n)$ the set of congruence classes of oriented $q$-spheres of $R^n$. All $n$-elements of $R^n$ are flat, while $(n - 1)$-spheres in $R^n$ are the subject of a well-known unproved conjecture.

**Theorem D.** $[g, n]$ and $(g, n)$ are commutative (additive) associative systems with a zero element which is the congruence class of flat elements or spheres, respectively.

The result is also true for $(n - 1)$-spheres in $n$-space, provided a suitable convention is adopted regarding orientation.

The operation of "addition" implied in Theorem D is obtained as follows in the case $[g, n]$:

Let $E^q, F^q$ be disjoint oriented $q$-elements of $R^n$. Let $g^q$ be a flat oriented $q$-element of $R^n$ such that

$E^q \cap g^q = \hat{E}^q \cap \hat{g}^q = e^{q-1}$, \quad a q - 1-element

$F^q \cap g^q = \hat{F}^q \cap \hat{g}^q = f^{q-1}$, \quad a q - 1-element

($\hat{E}^q$ denotes the boundary of $E^q$, etc.). Furthermore, let $e^{q-1}, f^{q-1}$ have flat $q$-dimensional neighborhoods in $E^q \cup F^q \cup g^q$, and let the orienta-
tions of $g^2$, $E^2$ and $g^2$, $F^2$ induce opposite orientations in $e^{q-1}, f^{q-1}$, respectively. Then $G^2 = E^2 \cup F^2 \cup g^2$ is an oriented $q$-element; and we prove that the congruence-class in $R^n$, $\gamma$ say, of $G^2$ depends only on the congruence classes $\alpha, \beta$ of $E^2, F^2$. We then define $\alpha + \beta = \beta + \alpha = \gamma$.

A similar construction is used in the case $(q, n)$, a slight further complication being due to the necessity of avoiding a possible "linking" of spheres.

The operation of removing a $q$-element from the interior of a flat $q$-element of a $q$-sphere of $R^n$ leads to a homomorphism

$$\tau: (q, n) \rightarrow [q, n].$$

The following semigroups are known to be zero:

$(q, n)$ if $2q + 2 \leq n$, by Theorem A.

$[q, n]$ if $2q + 1 \leq n$, by Theorems A and C, using the theory of section 5.

$(1, 2), (2, 3), [1, n], [3, 4]$.  

4. Results on embedding in $R^n$ can be applied to embedding in general $n$-manifolds. The definition of congruence in an $n$-manifold $M^n$ is the same as that given above in the special case $R^n$.

Let $P$ be a polyhedron embedded in an $n$-manifold $M^n$. We say that $P$ is locally embedded in $M^n$ if

(i) There is an $n$-element $E^n \subset M^n$ such that $P$ is in the interior of $E^n$;

(ii) $P$ does not disconnect $M^n$.

In particular, these conditions are satisfied if $P$ is an element in the interior of $M^n$, or if $P$ is a sphere in the interior of $M^n$ and $\dim P < n - 1$.

By $[P, M^n]$ we denote the pair consisting of a polyhedron $P$ locally embedded in an orientable manifold $M^n$ and the oriented manifold $M^n$.

Let $[P, M^n], [Q, N^n]$ be two such pairs, let $E^n \subset M^n$ and $F^n \subset N^n$ be $n$-elements such that $P, Q$ lie in the interior of $E^n$, $F^n$, respectively, and let $E^n, F^n$ have the orientations induced by $M^n, N^n$, respectively. If there is a $+PLO$

$$\theta: E^n \rightarrow F^n$$

such that

$$\theta P = Q,$$

then we say that the pairs $[P, M^n]$ and $[Q, N^n]$ are congruent, $[P, M^n] \equiv [Q, N^n]$.

**THEOREM E.** Let $P_i$ $(i = 1, 2)$ be locally embedded in the orientable $n$-manifold $M^n$, and $Q_i$ $(i = 1, 2)$ in the orientable $n$-manifold $N^n$. Let $P_i \equiv P_2$ in $M^n$ and $[P_i, M^n] \equiv [Q_i, N^n]$ $(i = 1, 2)$. Then $Q_1 \equiv Q_2$ in $N^n$.

**COROLLARY 1.** Let $M^n$, $N^n$ be equivalent manifolds. Then $[P, M^n] \equiv [Q, N^n]$ if and only if there is a $PLO \phi: M^n \rightarrow N^n$ such that $\phi P = Q$.

**COROLLARY 2.** $[P, M^n] \equiv [Q, M^n]$ if and only if $P \equiv Q$ in $M^n$.

Using this theorem it is easy to set up a one-one correspondence between
congruence classes of oriented polyhedra in an oriented $n$-manifold and in $R^n$. Using such a correspondence, for which a definite orientation of $R^n$ must be chosen once and for all, a locally embedded oriented $q$-sphere in an oriented $n$-manifold $M^n$ defines a unique element of $(q, n)$. This remark will be used in the next section.

5. Let $R^n$ be $R^n$ with a given orientation, and $M^n$ an oriented $q$-manifold of $R^n$, $q < n - 1$. Let $x \in M^n$; we can always find a simplicial subdivision $K$ of $R^n$ of which $L$, a subdivision of $M^n$, is a subcomplex and which has $x$ as a vertex. By $StLx$ we denote the star of $x$ in $L$ with the orientation induced by $M^n$ and by $LkLx$ the link of $x$, i.e., the set of simplices of $StLx$ which do not contain $x$, with the orientation induced by that of $StLx$. Similar notations are appropriate to $K$. Then, using the remark at the end of 4, $[StLx, R^n]$ defines an element $\sigma(x, M^n)$ of $[q, n]$ and $[LkLx, LkLx]$ defines an element $\lambda(x, M^n)$ of $[q - 1, n - 1]$ if $x \in M^n$ and of $(q - 1, n - 1)$ if $x$ is interior to $M^n$. The elements $\sigma(x, M^n)$ and $\lambda(x, M^n)$ are independent of the partitions used in their definition. $\sigma(x, M^n) = 0$ if and only if $\lambda(x, M^n) = 0$.

If $n = 2q$, it is a consequence of Theorem A that $\lambda(x, M^n) = 0$ unless $x$ is a vertex of some given partition of $M^n$; hence, for a finite manifold, there is at most a finite set of points $x \in M^n$ such that $\lambda(x, M^n) \neq 0$; also these points must be interior ones. Thus, with such a manifold we can associate uniquely a certain finite set of elements of $(q - 1, n - 1)$, called the "link type" of $M^n$.

**Theorem F.** The link type establishes a one-one correspondence between the elements of $[q, 2q]$ and (unordered) sets of the form $(\beta_1, \ldots, \beta_k)$ where $\beta_\infty(q - 1, 2q - 1)$, where the empty set stands for the zero element, and addition is given by $(\beta_1, \ldots, \beta_k) + (\beta_{k+1}, \ldots, \beta_l) = (\beta_1, \ldots, \beta_l)$.

6. The two theorems of this section are used in the proofs of the theorems on embedding:

Let $P$, $Q$ be polyhedra and each of $\phi_0, \phi_1 : P \rightarrow Q$ a PLO. If there is a PLO $\phi : P \times I \rightarrow Q \times I$, where $I$ denotes the unit interval and $P \times I$ is the topological product, such that $\phi(p, t) = (q, t)$, $\phi(p, 0) = (\phi_0 p, 0)$ and $\phi(p, 1) = (\phi_1 p, 1)$ for $p \in P$, $t \in I$, then we say that $\phi_0$ and $\phi_1$ are (piecewise onto) isotopic, and we write $\phi_0 \simeq \phi_1$.

**Theorem G.** Let $P$ be either an element or a sphere, and $\phi : P \rightarrow P$ a $+PLO$. Then $\phi \cong I$ (the identity map).

Let $M^n$ be an orientable $q$-manifold, and $E_i^q$, $i = 1, 2$, $q$-elements in its interior. Let $P \subset M^n - E_1^q - E_2^q$ be a polyhedron which does not disconnect $M^n$, and let $\psi : E_1^q \rightarrow E_2^q$ be a PLO which is orientation preserving in $M^n$.

**Theorem H.** There is a $+PLO$ $\phi : M^n \rightarrow M^n$ such that

$$\phi|_P = 1, \quad \phi|_{E_1^q} = \psi, \quad \phi \cong I.$$
In an earlier note\(^1\) in these PROCEEDINGS we have obtained an explicit Plancherel formula for a connected complex semisimple Lie group. The corresponding problem for a real group appears to be much more difficult due to the circumstance that two Cartan subgroups in it are in general not conjugate. In fact it seems likely that there is a close connection between classes of conjugate Cartan subgroups and the various "series" of unitary representations which occur in the Plancherel formula. The object of this note is to make a beginning in the study of this question by deriving an explicit Plancherel formula in the simple case of $2 \times 2$ real unimodular group.\(^2\)

Let $R$ be the field of real numbers and $G$ the group of all $2 \times 2$ real matrices with determinant 1. Then its Lie algebra $\mathfrak{g}_0$ consists of all matrices with trace zero. Put

\[
H = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}
\]

and

\[h_t = \exp tH, \quad n_s = \exp sX, \quad u_\theta = \exp \theta U \quad (t, s, \theta \in R).\]

Let $K$ and $A$ be the one parameter subgroups $G$ corresponding to $U$ and