ON THE STATISTICAL THEORY OF TURBULENT DIFFUSION*

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1. Introduction.—The fundamental equations of turbulent diffusion have been first established by G. I. Taylor. In his paper on diffusion by continuous movements, he considers the one-dimensional case of diffusion and finds the variance of the distance through which the particles have dispersed from a point-source. This variance is expressed as a function of the dispersion time and of two statistical characteristics of the turbulent field: the variance of the turbulent velocity and the Lagrangian correlation coefficient. Taylor's equations can be used as an approximation in the case of the three-dimensional turbulent diffusion from a continuous source in a fluid flow, when the turbulent velocities are small compared to the mean flow velocity. However, in many applications to fluid dynamics, meteorology, oceanography, chemistry, etc., turbulent velocities are of the same order of magnitude as the mean velocity or even of a larger order. The main purpose of this communication is to consider the turbulent diffusion in a field of homogeneous and isotropic turbulence when the turbulent velocities cannot be assumed small as compared to the mean velocity.

2. Fundamental Equations of Turbulent Diffusion.—Let us consider within an incompressible turbulent fluid at rest, of homogeneous and isotropic turbulence, a fluid particle which at an initial time $0$ is placed at the origin of the coordinate axes. Let $u(t)$, $v(t)$ and $w(t)$ be the $x$-, $y$- and $z$-components of the instantaneous velocity of the particle. After a time $t$, the coordinates of the particle are

$$
x = \int_0^t u(\alpha) d\alpha; \quad y = \int_0^t v(\alpha) d\alpha; \quad z = \int_0^t w(\alpha) d\alpha.
$$

The coordinates $x$, $y$ and $z$ are random variables and we shall investigate their variances as functions of the dispersion time $t$. The mean values of these coordinates $\bar{x} = \bar{y} = \bar{z} = 0$ and their variances are functions of the dispersion time, and since the turbulence is homogeneous and isotropic, $\bar{x^2} = \bar{y^2} = \bar{z^2} = \psi(t)$.

We shall determine $\psi(t)$ as a function of the turbulence characteristics. We find

$$
\bar{y^2} = [\int_0^t v(\alpha) d\alpha]^2 = \int_0^t \int_0^t v(\alpha_1)v(\alpha_2) d\alpha_1 d\alpha_2
$$

and introducing the Lagrangian correlation coefficient $R_h(h) = \frac{v(t)v(t+h)}{v^2}$,
\[ \overline{y^2} = \bar{v}^2 \int_0^t \int_0^t R_h(\alpha_2 - \alpha_1) \, d\alpha_1 \, d\alpha_2. \]

Taking into account that \( R_h(h) \) is an even function
\[
\overline{y^2} = \bar{v}^2 \int_0^t \int_0^t \alpha \, R_h(\alpha) \, d\alpha + \int_0^t \int_0^t -\alpha \, R_h(\alpha) \, d\alpha.
\]
and after inverting the order of integration and determining the new limits of integration, we find
\[
\overline{y^2} = 2\bar{v}^2 \int_0^t (t - \alpha) R_h(\alpha) \, d\alpha.
\]
This relation was found by a different method by J. Kampé de Fériet and is simpler for applications than the equation
\[
\overline{y^2} = 2\bar{v}^2 \int_0^t \int_0^t \alpha \, R_h(\alpha) \, d\alpha
\]
originally demonstrated by G. I. Taylor.

We shall consider two special cases: when the interval of time \( t \) is large and when it is small, as compared with the Lagrangian scale of turbulence \( L_h = \int_0^\infty R_h(\alpha) \, d\alpha. \)

**Large Dispersion Time:** When \( t \gg L_h \) equation (1) becomes
\[
(t \gg L_h) \quad \overline{y^2} = 2\bar{v}^2 L_h t - 2\bar{v}^2 \int_0^\infty \alpha R_h(\alpha) \, d\alpha,
\]
where the last term is constant. For very large dispersion times \( t \), this constant becomes small as compared with the first term on the right and
\[
(t \gg L_h) \quad \overline{y^2} = 2\bar{v}^2 L_h t.
\]

**Small Dispersion Time:** Consider now the case when \( t \ll L_h \). Expanding the correlation function in a power series, taking into account its evenness, and neglecting terms of \( h \) of an order higher than the second, we find \( R_h(h) \approx 1 - [h^2/(2\lambda_h^2)] \) where \( \lambda_h \) is called the Lagrangian micro-scale of turbulence. Substituting \( R_h(h) \) in (1), we obtain
\[
(t \ll L_h) \quad \overline{y^2} = \left( 1 - \frac{1}{12} \frac{t^2}{\lambda_h^2} \right) \bar{v}^2 t^2.
\]
When the ratio \( t/\lambda_h \) is small, the second term in the brackets may be neglected and
\[
(t \ll L_h) \quad \overline{y^2} = \bar{v}^2 t^2.
\]

**General Case:** When the dispersion time \( t \) cannot be considered small or large as compared with the Lagrangian scale of turbulence, the variation of \( y^2 \) as a function of \( t \) depends on the shape of the correlation curve. When this shape is known it may be possible to represent the correlation curve by a correlation function. Various representative functions can be used
for this purpose and the variation of \( y^2 \) as a function of \( t \) can then be found by using (1).

3. \textit{Mean Concentration Distribution.}—Let us consider a turbulent fluid at rest and let us assume that a large number of fluid particles are concentrated at an initial time zero at the origin of a set of rectangular axes. After an interval of time \( t \) these particles are dispersed in a cloud by the influence of the turbulent fluctuations. If the turbulence is homogeneous and isotropic, the cloud of particles is spherically symmetrical. In the preceding paragraph we have given the variance \( y^2 \) for such a cloud of particles as a function of the characteristics of turbulence. We shall now investigate the distribution of particles within a cloud. This distribution can be expressed by

\[
\text{Prob}[x < a < x + dx, y < b < y + dy, z < c < z + dz] = \psi(x, y, z) dx \, dy \, dz.
\]

The function \( \psi(x, y, z) \) is a probability density function and

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(x, y, z) dx \, dy \, dz = 1.
\]

We assume a three-dimensional Gaussian distribution and, therefore, a probability density function of the form

\[
\psi(x, y, z) = \frac{k^3}{\pi^{3/2}} \exp \left[ -k^2(x^2 + y^2 + z^2) \right].
\]  

(7)

When the variance \( y^2 \) for this distribution is determined, we find \( k^2 = 2y^2 \).

4. \textit{Instantaneous Point-Source of Dispersion in a Turbulent Fluid at Rest.}—Let us consider the case of a turbulent fluid at rest. At an initial moment 0, the particles are concentrated at the origin of the axes. The number (or quantity) of particles emitted at the initial moment is denoted by \( Q_0 \). \( \bar{s}_0(x, y, z, t) \) is the mean concentration of particles at a point \( x, y, z \), and after a dispersion time \( t \). The mean concentration is measured by the number (or quantity) of particles per unit of volume. We assume that \( Q_0 \) is very large (or that the particles are divisible) and, therefore, at the initial time \( \bar{s}_0(0, 0, 0, 0) = \infty \), \( \bar{s}_0(x, y, z, 0) = 0 \).

Applying (7), we find that after a dispersion time \( t \), the number of particles found in a volume element \( dx \, dy \, dz \) is equal to \( \bar{s}_0(x, y, z, t) \, dx \, dy \, dz \), with

\[
\bar{s}_0(x, y, z, t) = \frac{Q_0}{(2\pi y^2)^{3/2}} \exp \left[ -\frac{1}{2y^2} (x^2 + y^2 + z^2) \right].
\]  

(8)

In this equation \( y^2 \) is to be expressed as a function of \( v^2, R_a(h) \), and \( t \) by (1).

When we consider the dispersion of particles in a turbulent flow whose mean velocity \( \bar{u} = U \) is parallel to the \( x \)-axis, then the coordinate \( x \) in (8) should be replaced by \( (x - Ut) \).

5. \textit{Continuous Point-Source of Dispersion in a Fluid Flow.}—Consider now, in a fluid flow, the dispersion from a continuous point-source emitting steadily \( Q \) particles per unit of time. The origin of a set of rectangular
axes is placed at the point-source with the x-axis parallel to the direction of the mean velocity \( \bar{u} = U \). Due to the choice of axes, the components of the turbulent velocity are \( u' = u - U, \ v' = v, \ w' = w \). The mean concentration distribution for the time interval \( (t, t + dt_0) \) of those \( Q dt_0 \) particles which were emitted during the interval of time \( (t_0, t_0 + dt_0) \) can be represented by (8), if we replace \( S_0(x, y, z, t) \), \( Q_0 \) and \( x \), respectively, by \( S(x, y, z, t - t_0) dt_0, \ Q dt_0 \) and \( (x - Ut) \).

Taking the sum of the effects of the emission from \(-\infty \) till \( t \), we find that

\[
\bar{s}_{v}(x, y, z) = \int_{-\infty}^{t} S(x, y, z, t - \alpha) d\alpha = \int_{0}^{\infty} S(x, y, z, \beta) d\beta \tag{9}
\]

is independent of the dispersion time. \( \bar{s}_{v}(x, y, z) U dydz \) is the average number of particles which pass per unit of time through an element of area \( dydz \) of a plane perpendicular to the direction of the mean velocity \( U \).

From (8) and (9), we find the mean concentration flux

\[
\bar{s}_{v}(x, y, z) U = U \int_{0}^{\infty} \frac{Q}{(2\pi \bar{y}^2)^{1/2}} \exp \left\{ -\frac{1}{2\bar{y}^2} [(x - U\beta)^2 + y^2 + z^2] \right\} d\beta \tag{10}
\]

where \( \bar{y}^2 = \frac{y^2}{(2\pi \bar{y}^2)^{1/2}} \).

**Non-dimensional Mean Concentration Flux:** We shall now represent (10) in a non-dimensional form. With this representation it will be easier to examine the dependence of the mean concentration flux on the various parameters.

We introduce new coordinates \( \xi = x/UL_h, \ \eta = y/UL_h, \ \zeta = z/UL_h \). The Lagrangian correlation function is represented as a function of \( h/L_h \) by \( \mathfrak{B}_h(h/L_h) = R_h(h) \), and the intensity of turbulence is given by \( T = \sqrt{u'^2/U} = \sqrt{v'^2/U} = \sqrt{w'^2/U} \).

The mean concentration flux is represented as a function of the above coordinates by \( \sigma_v(\xi, \eta, \zeta) = U^2L_h^2[\bar{s}_{v}(x, y, z) U/Q] \), and \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma_v(\xi, \eta, \zeta) d\eta \ d\zeta = 1 \). With the non-dimensional notations, (10) becomes

\[
\sigma_v(\xi, \eta, \zeta) = \frac{1}{(2\pi)^{1/2} T^3} \int_{0}^{\infty} \frac{1}{\mathfrak{c}^4(\beta_1)} \exp \left\{ -\frac{1}{2T^2\mathfrak{c}^2(\beta_1)} \times \right\} \left\{ (\xi - \beta_1)^2 + \eta^2 + \zeta^2 \right\} d\beta_1 \tag{11}
\]

where \( \mathfrak{c}^4(\beta_1) = 2 \int_{0}^{\beta_1}(\beta_1 - \gamma_1) \mathfrak{B}_h(\gamma_1) d\gamma_1 \) represents \( \mathfrak{c}^4(\tau) \) for \( \tau = t/L_h = \beta_1 \).

We see immediately that the mean concentration distribution is a function of (in addition to the coordinates): (a) the shape of the Lagrangian correlation curve determined by the function \( \mathfrak{B}_h(h/L_h) \); (b) the intensity of turbulence \( T \) and (c) the product \( UL_h \) of the mean velocity by the Lagrangian scale of turbulence.

We shall consider two cases: when the distance from the emission source
x is large and when it is small compared to $UL_n$, i.e., when $\xi$ is, respectively, large and small compared to unity.

**Large Distances from the Source:** When the dispersion time is large, (4) can be applied and $t^2 = 2\tau$. Replacing $t^2(\beta_1)$ in (11) by $2\beta_1$, we find after integration

$$
\sigma_u(\xi, \eta, \xi) = \frac{1}{4\pi T^2\tau} \exp \left( \frac{\xi - r}{2T^2} \right)
$$

with $r = \sqrt{\xi^2 + \eta^2 + \xi^2}$.

The dispersion factor $\iota$ was determined neglecting the diffusion in the direction of the mean flow. The value of $\iota$ corresponds, therefore, to a one-dimensional case of diffusion. We now recompute $\iota$ taking into account the three-dimensionality of the dispersion wake. The new dispersion factor $\iota_d$ represents the standard-deviation $\sqrt{y_d^2}$ of the mean concentration measured in a plane perpendicular to the direction of the mean velocity along a diameter passing through the dispersion axis $Ox$. We have

$$
\iota_d^2 = \frac{1}{L_n^2} \frac{y_d^2}{v^2} = \frac{\int_0^\infty \eta^2 \sigma_u(\xi, \eta, 0) d\eta}{T^2 \int_0^\infty \sigma_u(\xi, \eta, 0) d\eta}
$$

Applying (12) and replacing in the two integrals the ratio $\eta/\xi$ by a hyperbolic sine and using the Schlöfi transformation, we find

$$
\iota_d^2 = 2\xi \frac{K_1(\xi/2T^2)}{K_0(\xi/2T^2)}
$$

Since the ratio of the first order Bessel function $K_1$ to the zero order Bessel function $K_0$ tends to one when their common argument tends to infinity, we have

$$
\lim_{t/2T^2 \to \infty} \iota_d^2 = 2\xi. \tag{14}
$$

At the limit, we have, therefore, a relation corresponding to the equation which G. I. Taylor has given for the one-dimensional case.

It should be noted that equation (13) is valid only when $\xi$ is much larger than unity. On the other hand, when $\xi/2T^2$ is very large, equation (13) is well represented by its asymptotic relation (14). Equation (13) will, therefore, be the most useful when at the same time both $\xi$ and $T$ are large. More particularly, equation (13) should be used when the turbulent velocities are of the same (or of a larger) order of magnitude as the mean velocity.

An approximate relation can be given for $\sigma_u(\xi, \eta, \xi)$ for the case when $\eta$ and $\xi$ are small compared with $\xi$. Developing $(\xi - r)$ in a series and
neglecting the high order terms, we find\(^8\) from (12)
\[
\sigma_u(\xi, \eta, \zeta) \approx \frac{1}{4\pi T^2} \frac{1}{\xi^2} \exp \left( -\frac{1}{4T^2} \frac{\eta^2 + \zeta^2}{\xi^2} \right)
\]  
(15)

**Small Distances from the Source:** When the distance from the source is sufficiently small, we find from (6) that \(\iota = \tau\). After replacing in (11) \((\beta_i)\) by \(\beta_i\), and performing the integration, we find\(^7\)
\[
\sigma_u(\xi, \eta, \zeta) = \frac{1}{(2\pi)^{1/4}} \frac{1}{r^2 T} \exp \left( -\frac{1}{2T^2} \right) \left[ 1 + \sqrt{\frac{\pi}{2T}} \times \right.
\exp \left( \frac{1}{2T^2} \frac{\xi^2}{r^2} \right) \text{erfc} \left( -\frac{1}{\sqrt{2T}} \frac{\xi}{r} \right) \left. \right] 
\]  
(16)

where \(r^2 = \xi^2 + \eta^2 + \zeta^2\) and \(\text{erfc}(X) = 1 - (2/\sqrt{\pi}) \int_0^X \exp(-\alpha^2) d\alpha\).

When the intensity of turbulence \(T\) is small compared to \((\xi/r)\), the second term in the brackets is large compared with the first term and neglecting the first, we find
\[
\sigma_u(\xi, \eta, \zeta) \approx \frac{1}{4\pi T^2} \frac{1}{r^2} \exp \left[ -\frac{1}{2T^2} \left( 1 - \frac{\xi^2}{r^2} \right) \right] \text{erfc} \left( -\frac{1}{\sqrt{2T}} \frac{\xi}{r} \right). 
\]  
(17)

For small values of \(\eta\) and \(\zeta\) as compared with \(\xi\) (and when \(T\) is small), we find the simple relation
\[
\sigma_u(\xi, \eta, \zeta) \approx \frac{1}{2\pi T^2} \frac{1}{\xi^2} \exp \left( -\frac{1}{2T^2} \frac{\eta^2 + \zeta^2}{\xi^2} \right).
\]  
(18)

It should be noticed that only with equation (16) do we find finite mean concentrations upstream of the dispersion origin.

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3 A random function depends on a parameter \(\omega\) chosen at random in a measure space \(\Omega\) with a measure equal to 1. The mean value of a function \(\Psi(x, y, z, t, \omega)\) is then given by \(\Psi(x, y, z, t) = \int_\Omega \Psi(x, y, z, t, \omega) d\mu\). In physical applications it is not possible to measure such mean values and time or space averages are used instead.

4 The only correlation coefficient used here is the Lagrangian time correlation coefficient, found when the turbulent velocity of the same fluid particle is considered as a function of time.


6 Various representative correlation functions of the form \(\exp[-k(|k|)]^n \phi(k)\) with \(n = 1\) and \(2\) are discussed in a paper prepared by the author in 1942 and published as *O.N.E.R.A. (Paris)*, *Rep. Tech.* 34 (1948).


8 Roberts, O. F. T., has given a similar relation [cf. p. 36 of Sutton, O. G., *Atmos-\)
To find his relation, Roberts uses the differential equation of turbulent diffusion \( \frac{\partial \delta u}{\partial t} + U \frac{\partial \delta u}{\partial x} = K \nu^2 \delta u \). It can be shown that the coefficient of eddy diffusion \( K \) is equal to \( \nu L_h \), if the molecular diffusion is neglected. The differential equation is valid only when \( t > L_h \). It is, however, possible [Frenkeli, F. N., Proc. Micrometeorological Symposium, International Union Geodesy and Geophysics, Brussels, Belgium, August, 1951] to define an “apparent coefficient of eddy diffusion” \( K_t = (\nu^2/t) \int_0^t (t - \alpha) \mathcal{R}_h(\alpha) \, d\alpha \), depending on the dispersion time \( t \), which, when used instead of \( K \) in the above differential equation, makes the equation valid for all values of \( t \).

# THE NUMBER OF SOLUTIONS OF CERTAIN EQUATIONS IN A FINITE FIELD

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Exact formulas for the number of solutions of certain special types of equations in a finite field have been obtained in recent papers by Hua and Vandiver,\(^1\) Faircloth,\(^2\) Faircloth and Vandiver;\(^3\) see also Ward.\(^4\) In the present note we state some additional theorems of this kind.

1. Let \( a_1, \ldots, a_k \) be rational integers such that \((a_1, \ldots, a_k) = 1\); then we can construct a matrix of integers \((a_{ij})\), \( i, j = 1, \ldots, k \), such that \( a_{ij} = a_i \) and \( \mid a_{ij} \mid = 1 \). Now if \( F \) is an arbitrary field, consider the equation

\[
\xi_1^{a_1} \cdots \xi_k^{a_k} = \alpha \quad (\alpha, \xi_t \in F) \tag{1.1}
\]

for fixed \( \alpha \). For \( \alpha = 0 \) the solution is evident; for \( \alpha \neq 0 \), no \( \xi_t = 0 \). We put

\[
\eta_t = \prod_{j=1}^{k} \xi_j^{a_{ij}} \quad (i = 1, \ldots, k),
\]

which is equivalent to

\[
\xi_t = \prod_{j=1}^{k} \eta_j^{b_{ij}} \quad ((b_{ij}) = (a_{ij})^{-1}).
\]

Thus (1.1) is equivalent to \( \eta_1 = \alpha, \eta_j \) arbitrary but \( \neq 0 \) for \( j = 2, \ldots, k \).

2. Now let \( F \) be the finite field \( GF(q) \), \( q = p^n \), and consider the equation

\[
\xi_1^{a_1} \cdots \xi_k^{a_k} = f(\xi_t, \ldots, \xi_r) \quad (\xi_t, \xi_j \in GF(q)), \tag{2.1}
\]

where \( f \) denotes an arbitrary polynomial with coefficients in \( GF(q) \). Applying the result of the previous paragraph we get