An essential component of the proof is the following lemma.

**Lemma C.** Given \( u \) subharmonic on \( F \) and \( f \) locally of type-BI, let \( U \) be defined on \( G \) as the upper function of \( \sup_{f(\theta)} u(\theta) \). If \( U < +\infty \), then \( U \) is subharmonic on \( G \).

A consequence of theorem B is that the classical theorem of Iversen may be extended to maps which are locally of type-BI. Lemma C admits a number of applications to other problems. By way of illustration we cite:

**Theorem D:** Let \( G \) have positive ideal boundary and let \( u \) be singular positive harmonic on \( G \). If \( f:F \rightarrow G \) is of type-BI, then \( u \circ f \) is singular on \( F \).

Theorem D leads to the following consequence for convergent infinite Blaschke products: the set of Fatou boundary values of modulus one of a convergent infinite Blaschke product is the unit circumference and each such Fatou boundary value of modulus one is attained at infinitely many points of the unit circumference.

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1 Frostman, O., Potentiel d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions. Lund Thesis (1935).


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**THE PARAMETRISATION AND ELEMENT OF VOLUME OF THE UNITARY SYMPLECTIC GROUP**

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We have given previously in these PROCEEDINGS\(^1\) a system of parameters for, and the complete element of volume (including the in-class factor as well as the well-known class factor) of, the \( n \)-dimensional rotation and unitary groups and we complete, in the present note, this problem for the classical groups by giving a system of parameters for, and the complete element of volume of, the \( 2k \)-dimensional unitary symplectic group.

The \( 2k \)-dimensional symplectic group, over the field of complex numbers, may be presented as the collection of \( 2k \times 2k \) matrices, \( X \), with complex elements, which are such that \( X^t I_{2k} X = I_{2k} \) where \( I_{2k} = \begin{pmatrix} 0 & -E_k \\ E_k & 0 \end{pmatrix} \).

Writing \( X \) as a \( 2 \times 2 \) block matrix \( \begin{pmatrix} A & C \\ B & D \end{pmatrix} \), whose elements are \( k \times k \) ma-
trices, \( X \) is symplectic if, and only if, \( B'A \) and \( D'C \) are symmetric and \( D'A - C'B = E_k \). Thus a 2 \( \times \) 2 matrix is symplectic if, and only if, it is unimodular. It is known that all symplectic matrices, for any value of \( k \), are unimodular but the proofs we have seen of this fundamental fact are unnecessarily complicated. In fact \( \begin{pmatrix} D' - C' \\ 0 \\ E_k \end{pmatrix} \begin{pmatrix} A & C \\ B & D \end{pmatrix} = \begin{pmatrix} E_k & 0 \\ B'D \end{pmatrix} \) so that \((\det D') (\det X) = \det D\). If, then, \( D \) is non-singular \( \det X = 1 \). If \( D \) is singular we observe that \( \begin{pmatrix} E_k & 0 \\ \epsilon E_k & E_k \end{pmatrix} \) is symplectic and unimodular, no matter what is the complex number \( \epsilon \), and we form the product \( \begin{pmatrix} E_k & 0 \\ \epsilon E_k & E_k \end{pmatrix} \), which is symplectic (being the product of two symplectic matrices). Since \( \epsilon C + D \) is singular for at most \( k \) values of \( \epsilon \) there exists an \( \epsilon \) for which \( \begin{pmatrix} A & C \\ \epsilon A + B & \epsilon C + D \end{pmatrix} \) is unimodular and this implies that \( \begin{pmatrix} A & C \\ B & D \end{pmatrix} \) is unimodular.

The unitary symplectic group of dimension \( 2k \) is the collection of \( 2k \times 2k \) matrices which are at once symplectic and unitary. Its typical matrix is of the form \( \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix} \) where \( B'A \) is symmetric and \( A^*A + B^*B = E_k \). It is a \((2k + 1)\)-parameter group, \( k \) of the parameters being class parameters and \( 2k^2 \) in-class parameters. Thus when \( k = 1 \), in which case the group is the unimodular unitary 2-dimensional group, we have 1 class and 2 in-class parameters; when \( k = 2 \), we have 2 class and 8 in-class parameters; when \( k = 3 \), we have 3 class and 18 in-class parameters and so on. It is natural, then, that the problem of determining the in-class factor of the element of volume of the group is a more serious one than the already solved problem of determining the class factor. Any unitary symplectic matrix \( X = \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix} \) may be transformed, by means of a unitary symplectic matrix \( V \), into diagonal form: \( V^*XV = \begin{pmatrix} D & 0 \\ 0 & \overline{D} \end{pmatrix} \). If the diagonal elements of \( D \) are \( e^{i\alpha_1}, \ldots, e^{i\alpha_k} \) the \( k \) angles \( \alpha_1, \ldots, \alpha_k \) (each of which may be taken to lie in the interval \( 0 \leq \alpha \leq \pi \)) may be taken as the class parameters of the group and the class factor of the element of volume of the group is well known to be

\[
\sin^2 \alpha_1 \ldots \sin^2 \alpha_k \prod_{p < q} \left( 1 - \cos(\alpha_p - \alpha_q) \right) \left( 1 - \cos(\alpha_p + \alpha_q) \right) d\alpha_1 \ldots d\alpha_k
\]

or the product of this by any positive constant.

In order to obtain a convenient system of in-class parameters we con-
sider the particular unitary symplectic $2k \times 2k$ matrices $(pq)$, each of which involves two parameters $\beta$ and $\tau$, $p$ and $q > p$ being any pair of distinct numbers from the set $1, \ldots, k$ or, else, $p$ being any number of this set and $q$ being $p + k$. The notation will be clear from the following examples for the case $k = 3$:

\[
(23) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & ce^{ir} - s & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & s & ce^{-ir} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & ce^{-ir} - s & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & s & ce^{ir} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & ce^{-ir} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

(14) = \begin{pmatrix}
cc^{ir} & 0 & 0 & -s & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}

where $c = \cos \beta$ and $s = \sin \beta$. We place a subscript under the parentheses $(pq)$ when we have to distinguish different pairs $(\beta, \tau)$; thus we obtain (23) by replacing $\beta$ and $\tau$ by $\beta_1$, $\tau_1$ in the formula above, and so on.

Then any $2 \times 2$ unitary symplectic matrix $X$ can be written as (12) $\delta$ where $\delta$ is a diagonal unitary symplectic $2 \times 2$ matrix; any $4 \times 4$ unitary symplectic matrix may be written as (12) (24) (12) (13) $\delta$, where $\delta$ is a diagonal unitary symplectic $4 \times 4$ matrix; any $6 \times 6$ unitary symplectic matrix may be written (12) (23) (36) (12) (23) (25) (12) (14) $\delta$, where $\delta$ is a diagonal unitary symplectic $6 \times 6$ matrix and so on. The factors preceding $\delta$ may be conveniently separated into sets of $1, 3, 5, \ldots, (2k - 1)$ each (counting from the right); thus, when $2k = 4$, we write \{(12) (24) (12)\} \{(13)\} $\delta$; when $2k = 6$ we write \{(12) (23) (36) (12) (23)\} \{(12) (25) (12)\} \{(14)\} $\delta$ and so on. Each of the angles $\beta$ lies in the interval $0 \leq \beta \leq \pi/2$ while each of the angles $\tau$ may lie in any one of the 4 quadrants. Turning to the formula $X = V(D^0_0) V^*$ we suppose $V$ factored in the manner described and observe that since $\delta$ commutes with $\begin{pmatrix}D & 0 \\ 0 & D\end{pmatrix}$ it may be omitted. Thus $V$ is the product of $1 + 3 + \ldots + (2k - 1) = k^2$ factors, each involving 2 parameters $\beta$, $\tau$, and we take these $2k^2$ parameters as the in-class parameters of the group.

The in-class factor of the element of volume of the group possesses the cumulative property (with respect to the dimension of the group) that the element of volume of the $(2k + 2)$-dimensional group contains as a factor the element of volume of the $2k$-dimensional group. The coefficient of $(d\beta) (d\tau)$, i.e., of the product of the differentials of the $k^2$ $\beta$'s and $k^2$ $\tau$'s, is a function of the $\beta$'s only and always contains $\sin 2\beta_1 \sin 2\beta_2 \ldots \sin 2\beta_k$, which we shall denote simply by $\Pi(\sin 2\beta)$, as a factor. For $k = 1$ the in-class factor of the element of volume is

$$dV_1 = (\sin 2\beta)d\beta d\tau$$
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(it being always understood that the element of volume may be multiplied by any convenient positive constant). For \( k = 2 \) it is

\[
dV_4 = (\cos^4 \beta) (\sin^2 \beta) \Pi (\sin 2\beta) (d\beta) (d\tau)
\]

there being 4 \( \beta \)'s and 4 \( \tau \)'s. For \( k = 3 \) it is

\[
dV_6 = (\cos^6 \beta) (\cos^4 \beta) (\sin^2 \beta \sin^4 \beta \sin^2 \beta) (\sin^2 \beta) \Pi (\sin 2\beta) (d\beta) (d\tau)
\]

there being 9 \( \beta \)'s and 9 \( \tau \)'s. For \( k = 4 \) it is

\[
dV_8 = (\cos^8 \beta) (\cos^6 \beta) (\cos^4 \beta) (\sin^2 \beta \sin^4 \beta \sin^6 \beta \sin^4 \beta \sin^2 \beta \sin^2 \beta_1) \times (\sin^2 \beta \sin^4 \beta \sin^2 \beta) \Pi (\sin 2\beta) (d\beta) (d\tau)
\]

there being 16 \( \beta \)'s and 16 \( \tau \)'s; and so on. In \( dV_{2k} \) the part involving the cosines starts out with \( \cos^{2k} \beta_1 (k-1)+1 \), each factor of this part being \( \cos^{2j} \beta_j (k-j)+1, j = k, k - 1, \ldots, 2 \). The part involving the even powers of the sines starts out with \( \sin^2 \beta_{2j} \), and consists of various factors corresponding to the described grouping of the factors of \( V \). For example, when \( k = 5 \), in which case we have 50 in-class parameters, the first (and largest) of these groups of factors is\( (12) (23) (34) (45) (5.10) (12) (23) (34) (45) \) and the corresponding part of the factor which involves the even powers of the sines is

\[
\sin^2 \beta_{24} \sin^4 \beta_{23} \sin^6 \beta_{22} \sin^8 \beta_{21} \sin^2 \beta_{19} \sin^4 \beta_{18} \sin^6 \beta_{17}.
\]

1 Murnaghan, F. D., these PROCEEDINGS, 38, 69–73, 127–129 (1952).

HEAT CONDUCTION ON ARBITRARY RIEMANNIAN MANIFOLDS

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Let \( M = M^n \) be an arbitrary orientable Riemannian manifold of class \( C^\infty \) and of dimension \( n \). We consider the exterior differential forms on \( M \), and we use the notation of de Rham\(^2\) in which the Laplace-Beltrami operator is the negative of the usual one. The scalar product of two \( p \)-forms \( \varphi, \psi \) is

\[
(\varphi, \psi) = \int_M \varphi \wedge^* \psi = (\psi, \varphi)
\]

while the Dirichlet scalar product is

\[
D(\varphi, \psi) = (d\varphi, d\psi) + (\delta \varphi, \delta \psi)
\]