*This investigation was supported by a grant from the National Foundation for Infantile Paralysis.

2 Adams, M. H., in Methods in Medical Research, Chicago, 1950.
4 Jacob, F., personal communication.
5 We wish to thank Ciba S.A., Basel, Switzerland, who put at our disposal the chlorohydrate of methyl bis-$\beta$-chloroethylamine, called Dichlorene, used in these experiments.
7 Hayes, W., J. Gen. Microb., 8, 72 (1953).

RELATIONS ON ITERATED REDUCED POWERS

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In this note we present the generalization of the relations on iterated squares to the case of iterated cyclic reduced powers of arbitrary prime period $p$. As in the case $p = 2$, the new relations are used to solve some particular problems.

Throughout this paper we will use the definitions and notation recently introduced by Steenrod.\(^2\)

1. For any complex $K$ and odd prime $p$, the cyclic reduced power operations are homomorphism $\phi^s$, ($s = 0, 1, \ldots$),

$$\phi^s: H^q(K; \mathbb{Z}_p) \rightarrow H^{q+2s(p-1)}(K; \mathbb{Z}_p)$$

They satisfy the following properties: $\phi^s f^* = f^* \phi^s$, where $f$ is a map of one complex into another; $\phi^0 = \text{identity}$; if $q = \dim u$ is even, $\phi^{s/2}u = u^p$ (in cup-product sense); $\phi^s u = 0$ when $s > q/2$.

As in the case of squares, an *iterated* cyclic reduced power is a composition of two or more of the $\phi^s$, e.g., $\phi^s \phi^s \phi^s$.

Let $\delta^*$ be the coboundary operator associated with the exact coefficient sequence $0 \rightarrow Z \rightarrow Z \rightarrow Z_p \rightarrow 0$. Our main result is the following

**THEOREM 1.1** *For all $0 \leq r < sp$ the iterated cyclic reduced powers satisfy the following set of relations*\(^6\)

\begin{align*}
(1.2) \quad & \phi^r \phi^s = \sum_{i=0}^{[r/p]} (-1)^{r+i} \left( \begin{array}{c}
(s - i)(p - 1) - 1 \\n0
\end{array} \right) \phi^{r+i-i} \phi^i, \\
(1.3) \quad & \phi^r \delta^* \phi^s = \sum_{i=0}^{[r/p]} (-1)^{r+i} \left( \begin{array}{c}
(s - i)(p - 1) - 1 \\n0
\end{array} \right) \delta^* \phi^{r+i-i} \phi^i + \\
& \sum_{i=0}^{[r-1/p]} (-1)^{r+i+1} \left( \begin{array}{c}
(s - i)(p - 1) - 1 \\n0
\end{array} \right) \phi^{r+i-i} \phi^i, \quad (\mod p),
\end{align*}
where \( \binom{n}{k} \) denotes the binomial coefficient with the usual conventions.

An induction argument based on (1.2) proves the following

**Theorem 1.4.** The set \( \{ \varphi^k \} \) with \( k = 0, 1, \ldots \), form a base in the sense that any other \( \varphi^r \) can be expressed as a sum of iterated cyclic reduced powers with exponents powers of \( p \).

For example,

\[
\varphi^r = \frac{1}{r!} (\varphi^1)^r \quad \text{for } 0 < r < p.
\]

\[
\varphi^{2p} = \frac{1}{2} (\varphi^p)^2 + \frac{1}{2} (\varphi^1)^{p-1} \varphi^p \varphi^1
\]

where \( (\varphi^j)^r \) means \( \varphi^j \) iterated \( r \) times.

Again, an induction using formula (1.2) proves

**Theorem 1.5.** The iterated powers of the type \( \varphi^{i_1} \ldots \varphi^{i_r} \) with \( i_1 \geq p i_2, \ldots, i_{r-1} \geq p i_r \), and \( c = i_1 + \ldots + i_r \), form an additive base for all iterated powers \( \varphi^{j_1} \ldots \varphi^{j_s} \) where \( j_1 + \ldots + j_s = c \).

2. For the particular values \( r = 1, s = 2^k - 1 \), formula (1.2) becomes

\[
\varphi^1 \varphi^{x^k-1} = 2^k \varphi^{x^k},
\]

therefore, if \( \dim u = 2^k + 1 \) we have

\[
(2.1) \quad u^p = \frac{1}{2^k} \varphi^1 \varphi^{x^k-1} u \quad \pmod p,
\]

where \( u^p \) is the \( p \)-power of \( u \) in the cup-product sense.

Let \( H(K) \) denote the integral cohomology ring of a complex \( K \). We say that \( H(K) \) is a truncated polynomial ring on \( u \) if \( H(K) \) is generated by the cup-product powers of \( u \) and each power is of infinite order. The height of \( u \) is the minimal integer \( n \) such that \( u^n = 0 \).

**Theorem 2.2.** If \( H(K) \) is a non-trivial truncated polynomial ring on \( u \) then \( \dim u = 2^k + 1 \). Moreover, if \( \dim u \geq 8 \) then the height of \( u \) is at most 3.

We will show how this theorem is implied by our relations on reduced powers. Let \( q = \dim u \). First, that \( q \) cannot be odd follows from the commutative law for cup-products. Now, if \( q \) is not a power of 2, then, because \( \{ \text{Sq}^n \} \) is a basis for squares,\(^1\) we have \( u \land u = \text{Sq}^q u = 0 \pmod 2 \), and this is a contradiction. Finally, suppose \( q = 2^k + 1 \). Using (2.1) for \( p = 3 \), we have

\[
u \land u \land u = (-1)^k \varphi^1 \varphi^{x^k-1} u \quad \pmod 3,
\]

and \( \varphi^{x^k-1} u = 3 \cdot 2^k + 1 - 4 \). Therefore \( u \land u \land u = 0 \pmod 3 \), unless \( 3 \cdot 2^k + 1 - 4 \) is a multiple of \( 2^k + 1 \). That is the case only if \( k = 0, 1 \).

Let \( S^{x-1} \) be a sphere bundle with \( S^{x-1} \) as fiber. Examples are known for the following forms of \( r \) and \( s \): \( r = s \); all \( r, s = 1 \); \( r = 2n, s = 2 \); \( r = 4n, s = 4 \); \( r = 16, s = 8 \).
Corollary 2.3. The other possible values of \( r \) and \( s \) for which \( S^{r-1} \) can be a sphere bundle with fiber \( S^{t-1} \) are of the form \( r = 2^k + 1 \) and \( s = 2^k, \) \((k \geq 4)\).

Proof: If \( S^{r-1} \) is fibered by \( S^{t-1} \), it follows from Gysin's sequence for sphere bundles that the integral cohomology ring of the base space \( B \) is a truncated polynomial ring, generated by the characteristic class \( u \) of dimension \( s \). Then \( r = ns \) for some integer \( n \) and \( \dim B = s(n - 1) \); therefore the height of \( u \) is \( n \). If \( n > 3 \), then 2.3 follows from 2.2. If \( n = 3 \) and \( f: S^{r-1} \rightarrow B \) is the projection, adjoin an \( r \) cell \( E' \) to \( B \) by means of \( f \), so that \( M = B \cup E' \) is a manifold. By duality \( H(M) \) is a truncated polynomial ring generated by \( u \) with height 4. This contradicts 2.2.

3. Our proof of relations (1.2), (1.3) is purely algebraic. The relations are obtained as homology relations on the symmetric group \( S_p^2 \) of degree \( p^2 \), and makes full use of the general definition for reduced power operations found recently by Steenrod. We will indicate briefly this method. Let \( G \) be a \( p \)-sylow group of \( S_p^2 \) and \( \theta: G \rightarrow S_p^2 \) the inclusion homomorphism. For each \( C \in H_i(G; Z_p) \) we have a reduced power operation. If \( u \in H^q(K; Z_p) \) then \( u^p/C \in H^{p^2-1}(K; Z_p) \).

To obtain the relations we first identify the operations induced by some cycles of \( H_i(G; Z_p) \) with sums of cyclic reduced powers. The relations are then obtained, according to the general principle of Steenrod, as elements on the kernel of \( \theta_*: H_i(G; Z_p) \rightarrow H_i(S_p^2; Z_p) \), i.e., if \( \theta_*(C_1 - C_2) = 0 \), then \( u^{p^2}/C_1 = u^p/C_2 \).

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1 Adem, J., these PROCEEDINGS, 38, 720–726 (1952).
3 I have heard that H. Cartan has obtained relations of the same type, using methods quite different from mine.