ERRATA: On Restricted Partitions and a Generalization of the Euler \( \phi \) Number and the Moebius Function

In the article of the foregoing title appearing in these PROCEEDINGS, 39, 963–968 (1953), the following corrections should be made:

(1) Relation (8), page 965, should read,

If \( n \) is odd,

\[
\sum_{d/n} (-1)^d \Phi(k, n/d) = \begin{cases} 0 & \text{if } (k, n) \neq n \\ -n & \text{if } (k, n) = n \end{cases}
\]

If \( n \) is even,

\[
\sum_{d/n} (-1)^d \Phi(k, n/d) = \begin{cases} n & \text{if } k = n/2 \\ 0 & \text{otherwise} \end{cases}
\]

(2) Relation (11), page 965, should read,

\[
\sum_{d/n} \left( \frac{d - 1}{\alpha} \right) (-1)^{d-\alpha-1} \Phi(k, n/d) \equiv 0 \mod. (n),
\]

where \( \alpha \) and \( \beta \) are the integral solutions of \((n/d)\alpha + \beta = s\) such that \(0 \leq \alpha < d\), \(0 \leq \beta < n/d\); and \(0 \leq s < n\).

C. A. NICOL
of contact. If every matrix \( \lambda A + \mu B \) has a multiple characteristic root it follows that \( C \) has a component which has to be counted double.

Detailed proofs and extensions to fields of finite characteristic will appear elsewhere.

into an odd number of unequal parts, none of which is larger than $n$. This then is an extension of the analogous problem arising in the use of the infinite product.

Also if $s$ is replaced by $-1$ we consider the following function.

$$(-1)^{n-1} F_{n-1}(-1, x) = \prod_{r=1}^{n-1} (1 + x^r).$$  \hspace*{1em} (4)

The coefficient of $x^s$ resulting from the expansion of this product is the number of partitions of $k$ as a sum of distinct positive integers none of which is larger than $n$. This may also be stated as the number of solutions of the equation $x_1 + 2x_2 + \ldots + (n-1)x_{n-1} = k$, where for $i = 1, 2, \ldots, (n-1)$, $x_i$ is either zero or unity.

The products (3) and (4) have been studied by Cauchy, T. Vahlen, von Sterneck, and others. In particular von Sterneck studied the case where the polynomial resulting from the expansion is reduced modulo a positive integer.

Fundamental in this investigation will be the use of the number

$$\Phi(k, n) = \frac{\varphi(n)}{\varphi(n/(k, n))} \mu(n/(k, n)),$$  \hspace*{1em} (5)

where $k$ and $n$ are positive integers and $(k, n)$ denotes the greatest common divisor of $k$ and $n$. If $n$ is a positive integer $\varphi(n)$ denotes as usual the number of positive integers less than $n$ and prime to it. ($\varphi(1) = 1$.) Also, for $n$ a positive integer, $\mu(n)$ is equal to $(-1)^\gamma$ where $\gamma$ is the number of distinct prime factors of $n$. ($\mu(1) = 1$.) Note that (5) reduces to $\mu(n)$ when $(k, n) = 1$ and $\varphi(n)$ when $(k, n) = n$. Although (5) appears more complicated than its constituents it will be shown that many of the principal theorems concerning it are hardly more complex than those involving the $\varphi$ or $\mu$ number alone.

The properties of the coefficients in the development of (1) are extensively used to obtain properties of (5) and vice versa.

We now state a number of theorems without proof. We hope to publish the proofs elsewhere.

In the following paragraphs the symbol $[x]$ will denote the largest integer contained in the real number $x$. Also the symbol $\Phi(a, b)$ will denote the number defined in (5) where $a$ and $b$ are positive integers.

We have if $k$, $r$, and $n$ are positive integers that

$$\sum_{(r, n) = 1} \exp(2\pi irk/n) = \Phi(k, n),$$  \hspace*{1em} (6)

where the range of $r$ is over all positive integers less than $n$ and prime to it. ($i^2 = -1.$)
We also have

**THEOREM 1.**

\[ \sum_{d|n} \Phi(k, d) = \begin{cases} n & \text{if } (k, n) = n. \\ 0 & \text{otherwise.} \end{cases} \]  

(7)

Similarly

\[ \sum_{d|n} (-1)^d \Phi(k, n/d) = \begin{cases} 0 & \text{if } (k, n) \neq n. \\ -n & \text{if } (k, n) = n, n \text{ even.} \\ 0 & \text{if } (k, n) = n, n \text{ odd.} \end{cases} \]  

(8)

We may also prove

**THEOREM 2.** Let \( \sigma_n(k) \) denote the sum of the divisors of \( k \) less than or equal to \( n \). Then,

\[ \sum_{s=1}^{n} \left\lfloor \frac{n}{s} \right\rfloor \Phi(k, s) = \sigma_n(k). \]  

(9)

In case \( k = 1 \) this becomes the well-known relation

\[ \sum_{s=1}^{n} \left\lfloor \frac{n}{s} \right\rfloor \mu(s) = 1. \]

Also if \( k \) is replaced by \( n! \) we have another known result

\[ \sum_{s=1}^{n} \left\lfloor \frac{n}{s} \right\rfloor \varphi(s) = n(n + 1)/2. \]

**THEOREM 3.** If \( \delta | n \), then

\[ \sum_{d|n} \left( \frac{dn/\delta}{d} \right) (-1)^d \Phi(k, \delta/d) \equiv 0 \pmod{n}. \]  

(10)

**COROLLARY.** If now \( p \) denotes an odd prime and \( \alpha \) is a positive integer, then

\[ \left( \frac{n}{p^\alpha} \right) \equiv \left( \frac{n/p}{p^{\alpha-1}} \right) \pmod{n}. \]  

(10a)

**THEOREM 4.**

\[ \sum_{d|n} R_s(d) \Phi(k, n/d) \equiv 0 \pmod{n}, \]  

(11)

where

\[ R_s(d) = \sum_{\alpha} \binom{d-1}{\alpha} (-1)^\alpha, \]

and this sum is over all integral solutions \( \alpha \) of the equation \((n/d)\alpha + \beta = s\) where \( 0 \leq s < n, 0 \leq \alpha < d, 0 \leq \beta < n/d. \)
Consider now the function $F_{n-1}(z, x)$ defined in (1). If $n > 1$ and the product is expanded as a polynomial in $x$, we may write

$$F_{n-1}(z, x) = \sum_{s=0}^{\bar{n}} P_s(z)x^s,$$

where $\bar{n} = n(n - 1)/2$ and $P_s(z)$ is a polynomial in $z$. Then we may define the polynomial $B_t(z)$ as

$$B_t(z) = \sum_{k=0}^{n_t} P_{zn+t}(z),$$

where $n_t = [(n - 1)/2 - t/n]$ and $1 \leq t \leq n$.

Then we may obtain

THEOREM 5. \textit{If $z$ is a number different from unity, then}

$$B_t(z) = \frac{1}{n(z-1)} \sum_{t=n/d}^{n} (z^{n/d} - 1)^d \Phi(t, n/d),$$

where $1 \leq t \leq n$ and $B_t(z)$ is defined in (13).

Denote the polynomial defined by the function $F_{n-1}(1, x)$ in (3) by

$$\sum_{s=0}^{\bar{n}} A_s x^s, \text{ where } \bar{n} = n(n - 1)/2.$$  

Also define the number $C_t$ by the relation

$$C_t = \sum_{k=0}^{n_t} A_{zn+t}, \text{ where } n_t = [(n - 1)/2 - t/n].$$  

(15a)

Then we obtain the following

THEOREM 6.

$$C_t = \Phi(t, n),$$

where $C_t$ is defined in (15a).

THEOREM 7.

$$\varphi(n) = \frac{1}{n} \sum_{i=1}^{n} C_i^2,$$

where $C_i$ is defined in (15a).

In view of theorem 6 we may write theorem 7 as

$$\varphi(n) = \frac{1}{n} \sum_{i=1}^{n} \Phi^2 (t, n).$$

(18)

An observation of possible interest may be made concerning theorem 7 if it is noted that $C_n = \varphi(n)$. Then (17) becomes a quadratic relation in $\varphi(n)$. Employing the quadratic formula we find
\[ \varphi(n) = (n \pm \sqrt{n^2 - 4G(n)})/2, \quad (19) \]

where \( G(n) = \sum_{i=1}^{n-1} C_i^2. \)

Except in the case when \( \varphi(n) = n/2 \) only one of the roots of (19) corresponds to \( \varphi(n) \). The significance of the remaining root has not been determined and would seem to be of interest.

If \( x \) is replaced by \( \exp(i\theta) \), where \( i^2 = -1 \), we obtain the following

**Theorem 8.**

\[ \Phi(t, n) = \frac{1}{2\pi} \int_0^{2\pi} \left\{ F_{n-1}(\exp(i\theta)) \sum_{k=0}^{n-1} \exp(-(kn + t)i\theta) \right\} d\theta, \quad (20) \]

where \( F_{n-1}(\exp(i\theta)) = \prod_{s=1}^{n-1} (1 - \exp(si\theta)) \) and \( n_t = [(n - 1)/2 - t/n] \).

Similarly for the numbers \( A_t \) defined in (15) we have

\[ A_t = \frac{1}{2\pi} \int_0^{2\pi} \left\{ F_{n-1}(\exp(i\theta)) \exp(-ti\theta) \right\} d\theta. \quad (21) \]

Furthermore the numbers \( A_t \) defined in (15) have the following properties:

\[ \sum_{s=0}^{n-2} |A_s| \geq n, \quad (22) \]

where \( \tilde{n} = n - (n - 1)/2. \)

A rather unusual property of these numbers is:

**Theorem 9.** If \( d|n-1 \), then

\[ \sum_{(s, n-1) = d} A_s = 0, \quad (23) \]

where \( 0 \leq s \leq n(n - 1)/2. \)

A by-product of these investigations is the following result: If \( p \) is an odd prime, the integral roots of the congruence

\[ 1 + \sum_{s=1}^{p-1} \Phi(s, p - 1)x^s \equiv 0 \pmod{p} \quad (24) \]

are the incongruent primitive roots modulo \( p. \)

If we consider the function defined in (4) we may obtain the following: If \( n \) is an odd positive integer and \( t \) is an integer such that \( 1 \leq t \leq n \), then

\[ B_t(-1) = \frac{1}{2n} \sum_{d|n} 2^d \Phi(t, n/d), \quad (25) \]

where \( B_t(z) \) is defined in (13).
Since $B_t(z)$ is a polynomial with integral coefficients the number $B_t(-1)$ is an integer.

In particular, if $n$ is odd and $t$ is less than $n$ and prime to it, then:

$$B_t(-1) = \frac{1}{2n} \sum_{d|n} 2^d \mu(n/d).$$

This number is a generalization of the Fermat Quotient, $(2^{p-1} - 1)/p$, where $p$ denotes an odd prime.

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5 Von Sterneck, Sitzungsber. d. Wiener Akad., 111, 1567 (1902); 113, 326 (1904); 114, 711 (1905).


7 Von Sterneck (P. Bachmann, *Ibid.*) introduced a function equivalent to $\Phi(k, n)$. He used it to obtain results concerning partitions modulo a positive integer. Employing this function he obtained a special case of theorem 1.

The number $\Phi(k, n)$ was used by R. Moller in the following result (*Math. Monthly*, 59, No. 4, 228 (April 1952)). If the numbers $g_d$ are all of the incongruent integers belonging to $d$ modulo $p$, $p$ being an odd prime and $d$ a divisor of $p - 1$, then for any $r$, $\sum g_d^r = \Phi(r, d) (\text{mod. } p)$

8 The sum $\sum_{(r, n)} \exp(2\pi i r k / n)$ is known as Ramanujan’s sum (cf., Hardy, G. H. and Wright, E. M., *Introduction to the Number Theory*, Oxford, 1938, pp. 55, 237). Another closed form for this sum was found previously by T. M. Apostol and D. R. Anderson and stated by them in an abstract in *Bull. Am. Math. Soc.*, 58, No. 5, 559 (1952). The form they found in our notation is $c \mu(b) \phi(a)/\phi(c)$ where $a = (n, k)$: $b = n/a$, and $c = (a, b)$. If $\mu(b) \neq 0$ we have the relation $\phi(n)/\phi(b) = c \phi(a)/\phi(c)$.
