The possibility of some form of dissociation is suggested by the harmonic relationship between the displacements of the components of the hydrogen lines, and the early appearance of the nebular lines in the spectra of these stars adds interest to considerations of this nature.


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**ON JACOBI'S EXTENSION OF THE CONTINUED FRACTION ALGORITHM**

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It has been known since Lagrange¹ that the regular continued fraction which represents a quadratic surd becomes periodic after a finite number of non-periodic partial quotients, and conversely, a regular continued fraction which becomes periodic after a finite number of non-periodic partial quotients is one root of a quadratic equation with rational coefficients. It is useless, therefore, to look for periodicity in regular continued fractions which represent cubic and higher irrationalities. To meet this difficulty Jacobi² undertook to extend the continued fraction algorithm as follows:

In the case of the ordinary continued fraction we are concerned with two series of numbers, $A_n, B_n$, (the numerators and denominators of the successive convergents) which are given by the recursion formulae

$$A_n = q_nA_{n-1} + A_{n-2}, \quad B_n = q_nB_{n-1} + B_{n-2},$$

with the initial values $A_0 = 0, A_{-1} = 1, B_0 = 1, B_{-1} = 0$. Jacobi considers three series of numbers, $A_n, B_n, C_n$, which are given by the recursion formulae

$$A_n = p_nA_{n-1} + q_nA_{n-2} + A_{n-3},$$

$$B_n = p_nB_{n-1} + q_nB_{n-2} + B_{n-3},$$

$$C_n = p_nC_{n-1} + q_nC_{n-2} + C_{n-3},$$

with the initial values:

$$(A_{-2}, A_{-1}, A_0) = (1, 0, 0), \quad (B_{-2}, B_{-1}, B_0) = (0, 1, 0),$$

$$(C_{-2}, C_{-1}, C_0) = (0, 0, 1),$$

Jacobi then chose the coefficients of the expansion, $p_n, q_n$, so that $A_n : B_n : C_n$ should approximate $1 : \theta : a + b\theta + c\theta^2$ where $\theta$ is the real root of a cubic equa-
tion, and showed that these coefficients sometimes become periodic. Bachmann\(^4\) showed later that with the coefficients determined by Jacobi's method periodicity will not always ensue. Berwick\(^4\) has obtained periodic expansion for cubic irrationalities but his algorithm differs from Jacobi's.

A closer study of Jacobi's expansion reveals many remarkable points of contact with ordinary continued fractions not hitherto observed. Instead of starting with a cubic irrationality and finding an expansion to fit it, we start with a periodic expansion and find associated with it a definite cubic irrationality.

The set of three numbers \((A_n, B_n, C_n)\) defined as above we call the \(n\)th convergent to the ternary continued fraction,

\[(p_1, q_1; p_2, q_2; p_3, q_3; \ldots)\]

The extension to four or more sets is obvious, and most of the theorems which follow hold also for quaternary and higher continued fractions. We reserve discussion of these, however.

The following theorem is of fundamental importance:

**Theorem I:** If \((A_\lambda, B_\lambda, C_\lambda)\) is the convergent of order \(\lambda\) in the ternary continued fraction \((p_1, q_1; p_2, q_2; p_3, q_3; \ldots)\), and \((A'_\lambda, B'_\lambda, C'_\lambda)\) the convergent of order \(\lambda'\) in the ternary continued fraction \((p'_1, q'_1; p'_2, q'_2; \ldots)\) then the convergent of order \(\lambda + \lambda'\), \((A''_\lambda + \lambda', B''_\lambda + \lambda', C''_\lambda + \lambda')\), of the ternary continued fraction \((p_1, q_1; p_2, q_2; \ldots; p_\lambda, q_\lambda; p'_1, q'_1; p'_2, q'_2; \ldots)\) may be obtained by the formulae.

\[
A''_\lambda + \lambda' = A_\lambda C'_\lambda + A_{\lambda - 1} B'_\lambda + A_{\lambda - 2} A'_{\lambda'},
B''_\lambda + \lambda' = B_\lambda C'_\lambda + B_{\lambda - 1} B'_\lambda + B_{\lambda - 2} A'_{\lambda'},
C''_\lambda + \lambda' = C_\lambda C'_\lambda + C_{\lambda - 1} B'_\lambda + C_{\lambda - 2} A'_{\lambda'}.
\]

More generally, the convergents of order \(\lambda + \lambda', \lambda + \lambda' - 1\) and \(\lambda - \lambda' - 2\) may be obtained by the rule for the multiplication of determinants, so that

\[
\begin{vmatrix}
A_\lambda & A_{\lambda - 1} & A_{\lambda - 2} \\
B_\lambda & B_{\lambda - 1} & B_{\lambda - 2} \\
C_\lambda & C_{\lambda - 1} & C_{\lambda - 2}
\end{vmatrix}
= \begin{vmatrix}
C'_{\lambda'}, & C'_{\lambda' - 2} & C'_{\lambda' - 2} \\
B'_{\lambda'}, & B'_{\lambda' - 1} & B'_{\lambda' - 2} \\
A'_{\lambda'} & A'_{\lambda' - 1} & A'_{\lambda' - 2}
\end{vmatrix}
\cdot
\begin{vmatrix}
A''_\lambda + \lambda', & A''_\lambda + \lambda' - 1, & A''_\lambda + \lambda' - 2 \\
B''_\lambda + \lambda', & B''_\lambda + \lambda' - 1, & B''_\lambda + \lambda' - 2 \\
C''_\lambda + \lambda', & C''_\lambda + \lambda' - 1, & C''_\lambda + \lambda' - 2
\end{vmatrix}
\]

The proof is by induction.

Consider now the purely periodic ternary continued fraction of period \(k\):

\[(p_1, q_1; p_2, q_2; \ldots; p_k, q_k)\]

Associated with this fraction is the following cubic equation which we shall call the *characteristic equation* of the ternary continued fraction:

\[
\begin{vmatrix}
A_k - 2 - \rho, & B_k - 2 & C_k - 2 \\
A_k - 1, & B_k - 1 - \rho & C_k - 1 \\
A_k, & B_k & C_k - \rho
\end{vmatrix} = 0
\]
Written at length this is
\[ \rho^3 - M\rho^2 + N\rho - 1 = 0 \]
where
\[ M = A_k - 2 + B_k - 1 + C_k \]
and
\[ N = A_k - 2 B_k - 1 - A_{k-1} B_k - 2 + B_k - 1 C_k - B_k C_{k-1} + A_{k-1} C_k - A_k C_{k-2}. \]

We have proved the following theorem concerning this equation:

**Theorem II:** The characteristic cubic of any periodic ternary continued fraction remains unaltered by any cyclic substitution of the partial quotients.

The proof is again by induction. Using this theorem we may prove the following recursion formulae:

**Theorem III:** For all integer values of \( n > 3k - 1 \) we have
\[
\begin{align*}
A_n &= MA_{n-k} - NA_{n-2k} + A_{n-3k}, \\
B_n &= MB_{n-k} - NB_{n-2k} + B_{n-3k}, \\
C_n &= MC_{n-k} - NC_{n-2k} + C_{n-3k}.
\end{align*}
\]

From Theorem III we see that the \( A \)'s, \( B \)'s and \( C \)'s are solutions of the following linear difference equation with constant coefficients:
\[ u_{x-3k} - Mu_{x-2k} + Nu_{x-k} - u_x = 0. \]

The theory of such equations is well understood (Boole's Finite Differences, p. 208). By referring to that theory we may write,
\[ A_n = \Sigma K_i x^i \quad (i = 1, 2, 3, \ldots 3k) \]
where \( K_1, K_2, \ldots K_3 \) are independent of \( n \) and \( x_1, x_2, \ldots x_m \) are the roots of
\[ x^{3k} - Mx^{2k} + Nx^k - 1 = 0, \]
and so are the \( k \)th roots of the roots of the characteristic cubic. This leads to the equation.
\[ A_n = \rho_1^{n/k} \Sigma P\omega^\nu + \rho_2^{n/k} \Sigma Q\omega^\nu + \rho_3^{n/k} \Sigma R\omega^\nu, \quad \nu = 1, 2, 3, \ldots k, \quad \omega^k = 1, \]
where \( P_\nu, Q_\nu, R_\nu, \) are independent of \( n \), and \( \rho_1, \rho_2, \rho_3 \), are the roots of the characteristic cubic. Similar equations hold for \( B_n \) and \( C_n \).

From this last result we obtain the remarkable theorem:

**Theorem IV:** If the characteristic cubic has one root, \( \rho_1 \), whose modulus is greater than the modulus of either of the other two roots then
\[
\lim_{M \to \infty} \frac{A_{n+k}}{A_n} = \lim_{M \to \infty} \frac{B_{n+k}}{B_n} = \lim_{M \to \infty} \frac{C_{n+k}}{C_n} = \rho_1.
\]

If the characteristic cubic has two imaginary roots whose common modulus is greater than the absolute value of the real root then the fractions \( A_{n+k}/A_n, B_{n+k}/B_n, C_{n+k}/C_n \) do not approach any limit as \( n \) increases beyond limit.
The characteristic cubic was obtained from the last three convergents at the end of the first period. If we form a cubic in the same way from the last three convergents at the end of the second period we get what we shall call the second characteristic cubic. Similarly for those of higher order. The following theorem holds:

Theorem V. The roots of the characteristic cubic of order \( \lambda \) are the \( \lambda \)th powers of the roots of the first characteristic cubic.

The proof is again by induction.

By a process similar to that by which Theorem IV was obtained we have also derived the following:

Theorem VI. If the characteristic cubic has one root \( \rho_1 \) whose modulus is greater than the modulus of either of the other two then the fractions \( B_n/A_n, C_n/A_n \) and \( C_nB_n \) approach for \( n \to \infty \) limits which are cubic irrationalities and are connected with the irrational number \( \rho_1 \) by linear fractional transformations. If the characteristic cubic has two imaginary roots whose common modulus is greater than the absolute value of the real root then these fractions do not approach any limit.

The actual equations connecting \( \rho_1 \) with \( \sigma_1 = \text{Lim} \ (B_n/A_n) \) and \( \sigma_2 = \text{Lim} \ (C_n/A_n) \) and \( \sigma_3 = \text{Lim} \ (C_n/B_n) \), when these limits exist are a little difficult to obtain, but turn out as follows:

\[
\begin{align*}
\sigma_1 &= (B_k\rho_1 + A_kB_k - \mu_1 - A_k B_k)/(A_k\rho_1 + A_k - 1 B_k - A_k B_k - 1) \\
\sigma_2 &= (C_k - 1 \rho_1 + C_k - 2 A_k - 1 - C_k - 1 A_k - 2)/(A_k - 1 \rho_1 + A_k C_k - 1 - A_k - 1 C_k) \\
\sigma_3 &= (C_k - 2 \rho_1 + C_k - 1 B_k - 2 - B_k - 1 C_k - 2)/(B_k - 2 \rho_1 + B_k C_k - 2 - B_k - 2 C_k).
\end{align*}
\]

The cubic equations satisfied by \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) may be obtained from these relations, but the algebraic work involved is tedious. It turns out that all the coefficients of the cubic in \( \sigma_1 \) are divisible by the determinant of the transformation connecting it with \( \sigma_1 \). It follows that the discriminant of the cubic in \( \sigma_1 \) is equal to the discriminant of the characteristic cubic multiplied by the square of the determinant of the transformation. Similar results hold for \( \sigma_2 \) and \( \sigma_3 \). The actual cubic equation satisfied by \( \sigma_1 \) is

\[
E\sigma_1^3 + F\sigma_1^2 + G\sigma_1 + H = 0
\]

where, \( E, F, G, H \) have the following values (we write for shortness \( A, B, C, \) for \( A_k, B_k, C_k; A_1, B_1, C_1 \) for \( A_1 - 1, B_1 - 1, C_1 - 1; \) and \( A_2, B_2, C_2 \) for \( A_2 - 2, B_2 - 2, C_2 - 2)\):

\[
\begin{align*}
E &= A^4C_1 + AA_1B_1 - AA_1C - A_1^2B \\
F &= A_1BB_1 + A_1BC - 2A_1A_2B + AA_1B_2 + AA_2B_1 - AA_2C - AB_1^2 + AB_2C \\
G &= -A_2^2B + A_2BB_1 + A_2BC + A_1BB_2 - 2ABC_2 - BB_2C + B^2C_1 + AA_2B_2 - 2AB_1B_2 + AB_2C, \\
H &= A_2BB_2 - BB_2C + B^2C_2 - AB_1^2.
\end{align*}
\]

(The equation has been freed from the factor \( A_1B_2 - AB (A_2 - B_1) - A_1A_2B_2 \) which is the determinant of the substitution connecting \( \sigma_1 \) and \( \rho_1 \).)
Theorems V and VI hold word for word when the ternary continued fraction has a finite number of non-periodic partial quotients.

Some progress has been made in the problem of finding a periodic ternary continued fraction which shall be the development of a given cubic irrationality, but the results are not yet in final form.


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**A CHARACTERIZATION OF JORDAN REGIONS BY PROPERTIES HAVING NO REFERENCE TO THEIR BOUNDARIES**

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Schoenflies¹ has formulated a set of conditions under which the common boundary of two domains will be a simple closed curve. A different set has been given by J. R. Kline.² Carathéodory³ has obtained conditions under which the boundary of a single domain will be such a curve. In each of these treatments, however, conditions are imposed 1) on the boundary itself, 2) regarding the relation of the boundary to the domain or domains in question. In the present paper I propose to establish the following theorem in which all the conditions imposed are on the domain R alone.

**Theorem.** In order that a simply⁴ connected, limited, two-dimensional domain R should have a simple closed curve as its boundary it is necessary and sufficient that R should be uniformly connected im kleinen.⁵

**Proof.** Suppose the simply connected domain R is connected im kleinen. Let M denote the boundary of R, that is to say the set of all those limit points of R that do not belong to R.

I will first show that M can not contain two arcs that have in common only one point, that point being an endpoint of only one of them. Suppose it does contain two such arcs EFG and FK with no point in common except F. Let α and β denote circles with common center at F, and with radii r₁ and r₂, respectively, such that r₁ > r₂ and such that E, K and G lie without α. By hypothesis there exists a positive number δₐ such that if X and Y are two points of R at a distance apart less than δₐ then X and Y lie together in a connected subset of R that lies wholly within some circle of radius r₂. Let γ denote a circle with center at F and with a radius less than one half of δₐ and also less than r₁ − r₂. It is clear that if two points of R are both within γ then they lie together in a connected subset of R that lies wholly within α. There