ON THE FLOW OF A PERFECT FLUID THROUGH A POLYGONAL NOZZLE. I*

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The problem of finding the steady flow of a two-dimensional perfect fluid through a prescribed nozzle into a medium of constant pressure leads to a question on conformal mapping which, if answered affirmatively, should yield a significant generalization of Riemann’s mapping theorem. Various special cases of this and of related theorems have been considered in some detail. The reader is referred particularly to the authors cited at the end of this note. The strongest known existence theorems on the problem considered\(^1\)\(^3\) are based on a hypothesis (i) that the prescribed configuration is symmetric with respect to an axis. The principal achievement of the present work consists in removing this restriction. Our results include those of Leray and Weinstein\(^1\)\(^1\) and overlap those of Garabedian and Spencer.\(^1\)\(^2\) We shall not here state them in full generality but shall rather indicate those new features which seem of most interest.

Consider a non-self-intersecting polygonal channel \(W\) of the type illustrated (Fig. 1). We seek a flow with uniform velocity at \(x = -\infty\), bounded in part by

![Figure 1](image)

\(W\) and in part by an interface of discontinuity, or free streamline \(\Lambda\), along which the pressure is constant. It should be noted that \(\Lambda\) is not prescribed but is to be determined by the condition of constant pressure.

We assert first the stability of a given flow, subject to the condition (ii) that all vectors from the origin, parallel to the walls \(W\) and directed in the sense of motion along \(W\) toward the separation points \(S, S'\), lie interior to a common half-plane.

**Theorem 1.** Let \(f(z) = \varphi + i\psi\) represent a flow through a polygonal channel \(W\) satisfying condition (ii). Then there is no infinitesimally neighboring flow through the same channel having streamlines distinct from those of the given flow.

The concept of the infinitesimally neighboring flow was first applied to this problem by Weinstein\(^2\) at the suggestion of Weyl.\(^2\)\(^0\) A precise definition appears in Friedrichs.\(^2\)

Theorem 1 has been proved by Weyl\(^2\)\(^0\) under additional assumptions, in particular a requirement that the neighboring flows have the same asymptotic width and direction. This requirement seems based on the necessity of obtaining a bounded
Dirichlet integral for the difference, or variation $\delta f$, of the flow functions considered. It prevents the application of the theorem to the problem of existence and uniqueness.

The present work uses a remark, due to Leray,\textsuperscript{10} on the behavior of $\delta f$ in the neighborhood of the separation points. The problem is then reduced to two alternatives, in the first of which a finite Dirichlet integral occurs and in the second of which the result follows from the Phragmén-Lindelöf theorem on harmonic functions. Essential use is also made of the work of Levi-Civita,\textsuperscript{12} of Weinstein,\textsuperscript{17, 18} and of Friedrichs.\textsuperscript{2}

Theorem 1 implies that a sufficiently small motion of the walls $W$ can be followed in a unique manner by a change in the flow function $f(z)$. It is easy to show that there is one and only one flow through a channel defined by two semi-infinite segments. Since channel walls $W$ satisfying condition (ii) can always be deformed continuously into parallel lines, the proof of existence and uniqueness of a flow through $W$ is made to depend on the possibility of following the entire motion of the walls by an appropriate deformation of $f(z)$. This, in turn, requires uniform estimates on the regularity of the flow throughout the motion. At present, we can give these estimates only under the additional assumptions (iii) that the nozzle is convex, i.e., all exterior angles at the intersection points of the wall segments have the sense indicated in the figure, and (iv) that the separation points $S$ and $S'$ can be separated by a horizontal line. Condition (ii) is then equivalent to a restriction on the total curvature $K$ of the nozzle, defined as the sum of all exterior angles at the intersection points of the wall segments.

**Theorem 2.** If channel walls $W$ satisfy conditions (iii) and (iv) and if $K < \pi$, then there is one and only one flow, bounded in part by $W$ and in part by free streamlines $\Lambda$ which are characterized by the requirement that the pressure along them be constant.\textsuperscript{‡}

The following corollary to Theorem 2 may be of interest: If channel walls $W$ satisfy conditions (i), (iii), and (iv) and if $K \leq \pi$, then there is one and only one flow through $W$, and the flow is symmetric with respect to the symmetry axis of $W$.\textsuperscript{‡}

I am indebted to Professor M. Morse for suggesting the field of study and to Professor A. Weinstein for suggesting the problem. A full exposition is being prepared.

**REFERENCES**

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\textsuperscript{†} The theorem referred to is stated on p. 395 of reference 3. I have been informed by Professor Garabedian that the requirements that the prescribed boundary be analytic and invariant under symmetrization are now known to be unnecessary. It should be of interest to determine whether the conceptually appealing variational principle used by these authors can be applied to the asymmetric configuration considered in the present note.

\textsuperscript{‡} The flow is unique only in a suitably restricted class of competing flows. It is sufficient to require that the angles made with the $x$-axis by the streamlines of the flow remain bounded in magnitude. This excludes, in particular, the possibility of spiral motion.

\textsuperscript{1} U. Cisotti, *Idrodinamica Piana* (Milan, 1921).

ON THE FLOW OF A PERFECT FLUID THROUGH A POLYGONAL NOZZLE. II*

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In a related note the flow of a two-dimensional perfect fluid through an asymmetric polygonal nozzle has been considered, and an existence and uniqueness theorem has been stated subject to hypotheses of convexity and of a limitation on the total curvature $K$ of the channel walls. (For appropriate definitions, references to pertinent literature, and orientation the reader is referred to the cited note.)

The present work concerns the flow of a two-dimensional perfect fluid through a symmetric, convex polygonal nozzle; however, no restriction is made on the total curvature of the walls, which is required merely to be finite. Also, self-intersections of a certain type are seen to be permissible. The flows considered may correspondingly be situated on multiply covered regions of the plane.

Definition: A polygonal, symmetric, convex channel wall $W$ will be called admissible provided that it admits a continuous deformation into a channel consisting of two straight parallel walls in such a way that (a) the deformation process is symmetric with respect to the axis of symmetry of $W$, (b) there is a fixed $\epsilon > 0$ such that all exterior angles remain nonnegative and smaller than $\pi - \epsilon$, (c) the lengths of all wall segments remain bounded away from zero, and (d) the separation points do not meet the channel walls during the deformation.

A channel wall that is not admissible will be termed inadmissible. Examples of admissible and of inadmissible walls are shown in Figure 1. Only the upper wall and line of symmetry are shown. It is of course possible to give a purely geometrical definition of admissibility. It is easily seen that for inadmissible channel walls there can in general be no nonsingular flow of the type sought.