ON THE FLOW OF A PERFECT FLUID THROUGH A POLYGONAL NOZZLE. II*

By Robert Finn

UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES

Communicated by Marston Morse, August 11, 1954

In a related note\(^3\) the flow of a two-dimensional perfect fluid through an asymmetric polygonal nozzle has been considered, and an existence and uniqueness theorem has been stated subject to hypotheses of convexity and of a limitation on the total curvature \(K\) of the channel walls. (For appropriate definitions, references to pertinent literature, and orientation the reader is referred to the cited note.)

The present work concerns the flow of a two-dimensional perfect fluid through a symmetric, convex polygonal nozzle; however, no restriction is made on the total curvature of the walls, which is required merely to be finite. Also, self-intersections of a certain type are seen to be permissible. The flows considered may correspondingly be situated on multiply covered regions of the plane.

**Definition:** A polygonal, symmetric, convex channel wall \(W\) will be called **admissible** provided that it admits a continuous deformation into a channel consisting of two straight parallel walls in such a way that (\(\alpha\)) the deformation process is symmetric with respect to the axis of symmetry of \(W\), (\(\beta\)) there is a fixed \(\varepsilon > 0\) such that all exterior angles remain nonnegative and smaller than \(\pi - \varepsilon\), (\(\gamma\)) the lengths of all wall segments remain bounded away from zero, and (\(\delta\)) the separation points do not meet the channel walls during the deformation.

A channel wall that is not admissible will be termed **inadmissible**. Examples of admissible and of inadmissible walls are shown in Figure 1. Only the upper wall and line of symmetry are shown. It is of course possible to give a purely geometrical definition of admissibility. It is easily seen that for inadmissible channel walls there can in general be no nonsingular flow of the type sought.
THEOREM 1. Let \( f(z) = \varphi + i\psi \) be the flow function of a symmetric flow bounded in part by admissible channel walls \( W \) and in part by lines \( \Lambda \) of discontinuity along which the pressure is constant. Then there is no infinitesimally neighboring flow, symmetric or asymmetric, through the same channel and having streamlines distinct from those of the given flow.

An example of an infinitesimal disturbance given by Weyl\(^2\) contains an error. The full proof of Theorem 1 is in its present form rather technical and will not be indicated here. For the case of symmetric disturbances, the theorem follows from an elementary property of harmonic functions.

![Figure 1](image-url)

(a) ![Admissible walls](image-url)

(b) ![Inadmissible walls](image-url)

**LEMMA.** Let \( h(x, y) \) be harmonic in the strip \( S: 0 < y < 1 \), and continuously differentiable on every compact subset of the closure of \( S \) except at the point \((0, 1)\), near which it is assumed that \( h_x^2 + h_y^2 \) is bounded. We suppose that \( h = 0 \) on the semi-infinite segment \( [y = 1, x \leq 0] \) and that \( h_y = ph \) on \( [y = 1, x \geq 0] \), where \( p \) is a non-increasing function of \( x \), \( 0 < x < \infty \), \( p \neq \) constant. We assume further that \( h_x \to 0 \) uniformly in \( y \) as \( x \to -\infty \), that \( h_y \to 0 \) uniformly in \( y \) as \( x \to +\infty \), and that \( h_x = 0 \) on \( [y = 0] \). Then \( h(x, y) \) vanishes identically.

The lemma is obtained as a consequence of a form of Green's identity, due to Levi-Civita.\(^3\)

Elementary estimates lead to an existence and uniqueness theorem, which we formulate as a theorem on conformal mapping.

**THEOREM 2.** For any prescribed admissible channel walls \( W \) there is a simply connected, unramified (but perhaps multiply covered) domain \( D \) bounded in part by \( W \) and in part by a symmetric continuum \( \Lambda \) joining to the separation points of \( W \), and a function \( f(z) \) defined in \( D \), such that \( f(z) \) maps \( D \) in a 1-1 conformal manner onto a strip of unit width and such that \( |f'(z)| \) approaches the same finite value \( \mu \) as \( z \to \Lambda \) in any manner. The continuum \( \Lambda \) is uniquely determined by the prescribed walls \( W \) among all symmetric continua joining to the separation points. For this \( \Lambda \), the function \( f'(z) \) (and hence also the constant \( \mu \)) is unique.\(^4\)

It seems natural to expect that the theorems of this and of the preceding note will be found to be special cases of a general property of conformal mapping. In this
respect we remark the recent work of Beurling. A detailed exposition is in preparation.

* This work was sponsored in part by the United States Air Force through the Office of Scientific Research. Part of the work was performed while the author was at the University of Maryland, and part while he was at Stanford University.

1 R. Finn, these PROCEEDINGS, 40, 983–85, 1954.
3 T. Levi-Civita, Atti reale Ist. veneto sci., 64, 1655, 1905.
4 The mapping is unique only in a suitably restricted class of mappings. (Cf. Finn, op. cit., n. †.)

HARMONIC MAPPINGS

BY F. B. FULLER

DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY

Communicated by S. Lefschetz, April 29, 1954

If $f$ is a mapping from one manifold to another, it is possible under suitable restrictions to associate an energy $E(f)$ with $f$. Any deformation of $f$ which increases the topological irregularity, as by putting in folds and wrinkles, will in general increase the energy $E(f)$. Hence those mappings with minimum energy in the homotopy class of $f$ might be expected to have considerable topological regularity and so provide a natural means of normalizing $f$ in homeomorphism and isotopy problems.

The method of defining $E(f)$ is illustrated by the following example. Let $f_0$ be a mapping from a circle $C_1$ to another circle $C_2$. Now regard the circle $C_1$ as being made of a uniform elastic material in which the tension is proportional to the dilatation, but which offers no resistance to compression, so that the only position of no tension is for $C_1$ to collapse to a point. The mapping $f_0$ constrains $C_1$ to lie in a certain configuration on $C_2$. If $C_1$ is now allowed to move over $C_2$ subject to forces caused by tension, it will leave its initial position $f_0$ and will approach a position of equilibrium represented by the mapping $f_1(x) = dx + c$, where $x$ and $f(x)$ are central angles, $c$ is a constant, and the integer $d$ is the topological degree of $f_1$ and hence also of $f_0$, since $f_0$ and $f_1$ are homotopic. If $f_0$ is of class $C^2$ (has two continuous derivatives), then, following the usual discussion of the vibrating string, $\frac{\partial^2 f_0}{\partial x^2}$ is the force density at a point of $C_1$ due to tension and

$$E(f_0) = \frac{1}{2} \int_0^{2\pi} (\frac{\partial f_0}{\partial x})^2 \, dx$$

is the energy of deformation of $f_0$. For the equilibrium mapping $f_1$ the force density vanishes and the energy is minimized.

The general definition of the energy and force density associated with a mapping is obtained by generalizing the above discussion for two circles to higher dimensions. Let $M^p$ be a compact orientable analytic manifold with co-ordinates $x^1, x^2, \ldots, x^p$ and an analytic metric $g_{ij}$. Similarly, let $N^q$ be a compact analytic manifold with co-ordinates $y^1, y^2, \ldots, y^q$ and an analytic metric $h_{\lambda\mu}$. Let $f$ be a mapping of class