and, since

\[ *R_1, \ldots, s^n = R_{s+1}, \ldots, r^n - s^n, \]

we have the further duality result

\[ H^n(M; C_1 \cup \ldots \cup C_s; R^e) = H^n - q(M; C_{s+1} \cup \ldots \cup C_r; R). \] (4.4)

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**ERGODIC PROPERTY OF THE BROWNIAN MOTION PROCESS**

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Let \( X(t) \), \( 0 \leq t < \infty \), denote a separable Brownian motion (Wiener) process on the line: \( X(0) = 0, X(t) \) is continuous for all \( t \) with probability 1, and for any \( t_0 < t_1 < \ldots < t_n \) the random variables \( X(t_j) - X(t_{j-1}), j = 1, \ldots, n \), are independent and normally distributed with zero means and variances \( t_j - t_{j-1} \). Kallianpur and Robbins\(^1\) considered ratios of the form \( \int_0^T f(X(t)) \, dt / \int_0^T g(X(t)) \, dt \) as \( T \to \infty \) and proved that if \( f(x) \) and \( g(x) \) are bounded and summable with \( \hat{f} = \int_\mathbb{R} f(x) \, dx, \hat{g} = \int_\mathbb{R} g(x) \, dx \neq 0 \), then the above ratio tends in probability to \( \hat{f}/\hat{g} \). (They also proved a similar result for the two-dimensional Brownian motion process.) We shall consider here the question of whether the limit exists with probability 1. Robbins\(^2\) stimulated interest in this type of problem in considering the equidistribution of sums of independent random variables and of certain stochastic processes, including the Wiener process. Harris and Robbins\(^3\), utilizing ergodic theory of spaces with infinite measures, obtained results like those of the present note for recurrent Markov chains with discrete time parameter. Chung\(^4\) proved, independently, the results of Harris and Robbins for the special case of Markov chains with denumerably many states. The proof used in this note will be a generalization of the one used by Chung, which in turn is based on an idea of Doeblin.\(^5\)

Let \( \tau \) be the random variable equal to min \( \{ t : X(t) = 1 \} \), and let \( \mu(A, \tau) \) denote the Lebesgue measure of the set \( \{ t : X(t) \in A, t \leq \tau \} \), where \( A \) is any Borel set. (Since \( X(t) \) is measurable with respect to the product space \( \Omega \times T \), where \( \Omega \) is the space of all sample functions \( X(t) \) and \( T \) is the set of nonnegative real numbers \( t \),
the set \( \{ t: X(t) \in A, t \leq \tau \} \) is Lebesgue measurable. Let \( \mu(A) = E\mu(A, \tau) \). We now prove Lemma 1.

**Lemma 1.** If \( A \) is any bounded Borel set, then

\[
\mu(A) = 2\int_0^\infty \frac{w(x, t)}{2\pi t} dt.
\]

**Proof:** Let \( Y(t) = 1 \) if \( X(t) \in A, t \leq \tau \) and \( Y(t) = 0 \) otherwise. It is easily verified that \( Y(t) \) is measurable with respect to the product space \( \Omega \times T \). Then we have

\[
\mu(A, \tau) = \int_0^\infty Y(t) dt.
\]

Taking expected values, where \( P \) denotes the probability measure over \( \Omega \), we have

\[
\mu(A) = \int_0^\infty E[Y(t)] dt = \int_0^\infty P_\tau \{ X(t) \in A, t \leq \tau \} dt
\]

where \( w(x, t) \) is the density function in \( x \) of the event that simultaneously \( X(t) = x \) and \( t \leq \tau \). The various interchanges of integrals carried out in (2) are justified, since the integrand is always nonnegative. By applying the principle of symmetry, it is seen that

\[
w(x, t) = \frac{1}{\sqrt{2\pi t}} (e^{-x^2/2t} - e^{-(x - t)^2/2}) \quad \text{for } x \leq 1
\]

and zero otherwise. It can be shown by differentiating under the integral sign or by direct evaluation that

\[
\int_0^\infty w(x, t) dt = \begin{cases} 2 & \text{for } x \leq 0, \\ 2(1 - x) & \text{for } 0 < x \leq 1, \end{cases}
\]

and zero otherwise. The lemma follows from (2) and (4).

Let \( \tau^* = \min \{ t > \tau: X(t) = 0 \} \) and \( \mu^*(A) = E\mu(A, \tau^*) \), where \( \mu(A, \tau^*) \) is the Lebesgue measure of the set \( \{ t: X(t) \in A, t \leq \tau^* \} \). It follows from Lemma 1, symmetry and the independent increments property of the Brownian motion process that

\[
\mu^*(A) = 2m(A),
\]

where \( m \) is the Lebesgue measure.

**Lemma 2.** If \( f(x) \) is any real-valued Borel measurable function, then

\[
E\int_0^{\tau^*} f(X(t)) dt = 2f.
\]

**Proof:** Since \( f(x) \) is Borel measurable, \( f(X(t)) \) is Borel measurable with probability 1, and therefore it makes sense to speak of the integral (6). Let \( A \) be any Borel set, and suppose \( f(x) = 1 \) if \( x \in A \), and zero otherwise. Then, from (5), we have

\[
E\int_0^{\tau^*} f(X(t)) dt = 2m(A).
\]

Lemma 2 follows by the usual approximation procedures.
THEOREM. If \( f(x) \) and \( g(x) \) are any two real-valued Borel measurable functions, summable over the real line \(-\infty < x < \infty\), and if \( g \neq 0 \), then with probability 1

\[
\lim_{T \to \infty} \frac{\int_0^T f(X(t)) \, dt}{\int_0^T g(X(t)) \, dt} = \frac{\bar{f}}{\bar{g}}
\]

Proof: Consider first the case where \( f(x) \leq 0 \) for all \( x \). Let \( t_1 = \min \{ t : X(t) = 1 \} \), \( t_2 = \min \{ t > t_1 : X(t) = 0 \} \), \( t_3 = \min \{ t > t_2 : X(t) = 1 \} \), \( t_4 = \min \{ t > t_3 : X(t) = 0 \} , \ldots \). Since the probability is 1 that \( \tau \) is finite, it follows that, with probability 1, the sequence \( \{ t_i \} \) is denumerable. Let \( Z_i = \int_{t_{i-1}}^{t_i} f(X(t)) \, dt \). Then the random variables \( Z_i \) are independent and identically distributed.

Let \( K(T) \) be the largest even number such that \( t_{K(T)} < T \). Then

\[
\sum_{i=1}^{K(T)} Z_i \leq \int_0^T f(X(t)) \, dt \leq \sum_{i=1}^{K(T) + 1} Z_i \tag{8}
\]

Since \( \Pr \{ \lim_{T \to \infty} K(T) = \infty \} = 1 \), it follows from the strong law of large numbers, and (6) that

\[
\Pr \left\{ \lim_{T \to \infty} \frac{\int_0^T f(X(t)) \, dt}{K(T)} = 2f \right\} = 1. \tag{9}
\]

If \( f(x) \leq 0 \), the same argument holds. Thus (9) holds for functions taking on both positive and negative values, since any function can be represented as the sum of two functions each of constant sign. The same argument holds for \( g(x) \). Hence, by taking ratios (provided that \( \bar{g} \neq 0 \)), the theorem follows.

It seems likely that the theorem proved by Kallianpur and Robbins\(^1\) for the two dimensional Brownian motion process also holds with probability 1, but the method of proof used here does not seem to carry over. The method will, however, carry over to strongly continuous separable Markov processes having the property that the first passage time between any two distinct points is finite with probability 1 and that \( \mu^*(A) \) is finite for all bounded Borel measurable sets. If the process \( X(t) \) is such that for any \( A \) there exist numbers \( T \) and \( T' \) for which

\[
\inf_{x \in A \cap \{ x \leq 1 \}} \Pr \{ X(T) \leq 1 \mid X(0) = x \} > 0
\]

and

\[
\inf_{x \in A \cap \{ x > 1 \}} \Pr \{ X(T') \leq 0 \mid X(0) = x \} > 0
\]

then \( \mu^*(A) \) will be finite.

It is interesting to note that (5) has its counterpart in the symmetrical random walk. It follows from a theorem proved by the author\(^7\) and was pointed out by Chung\(^4\) that the expected number of times any given state is visited before returning to the origin is equal to one.

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ON THE POSSIBILITY OF ELECTROMAGNETIC SURFACE WAVES

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1. Introduction.—The question of electromagnetic surface waves propagating along a plane of discontinuity between two different media was the subject of some diversity of opinion. It is related to the equally unsettled question about the merits of the conflicting solutions given by A. Sommerfeld and H. Weyl for the field of an electric dipole oscillating at the surface of a plane earth. A few remarks about this relation will be made in section 5, but its complete discussion will be postponed until a later date, and the present paper will be devoted to the analysis of the independent surface wave. It will be well, therefore, to define what is meant by this term. It is a matter of common knowledge that an ordinary plane wave, falling under an angle on a surface of discontinuity and suffering total reflection in the process, produces in the second medium an inhomogeneous wave with propagation parallel to the surface of discontinuity. But this may be called a dependent surface wave because it is only an adjunct of the space waves in the first medium and would not exist without them. On the other hand, an independent surface wave is one which consists of two inhomogeneous waves—one in each medium—running along the dividing surface. Its intensity is appreciable only in a thin layer along the surface of discontinuity and decreases exponentially to both sides.

On the whole, the results of the analysis presented in the following sections are negative: an independent surface wave is supported only by a medium with peculiar properties that are hardly available in nature (sec. 4); therefore, it is only of theoretical interest. The whole investigation hinges on the consideration of the physical constants of the materials involved. Let the complex dielectric constants of the first and second medium be noted, respectively, by

\[ k^2 = \kappa = \varepsilon + i\sigma, \quad k'^2 = \kappa' = \varepsilon' + i\sigma'. \]  

(1)

In every known medium the constant \( \sigma \) (which is proportional to the conductivity) is positive, and this fact has a decisive influence on the results. Moreover, it is arbitrary which medium is chosen as the primed; therefore, the convention