compute, for that instant, the distributions $P_{N_1}$, by averaging the $P_{N_j}$ for all particle distributions whose five distributions $D_N^j(t_0; R)$ coincide with the given one for each $R_j, j = 1, \ldots, Q_N$. This problem is simplified to that of computing the equilibrium distribution of $\nu$ particles in a cell $R$ with given values of $\rho$ and $\epsilon$. This reduces to the problem of computing ratios of the measures of sets on certain manifolds. This is a rather standard type of problem but is too long to describe here. The limiting process is carried through, allowing $\nu \to \infty$, and the results are seen to be of the form (18), in which the $q_i$ satisfy equation (20).

The ordinary hydrodynamic equations, without viscosity or heat conduction, follow easily in the usual way from the limiting equations in section 5, above; expressions are obtained for the pressure $p$ and for $\epsilon$ in terms of $\Phi$, $\rho$, and $A$. The analysis of the procedure in the immediately preceding paragraph yields further information about the equation of state "in the large" and introduces the entropy function in a natural way.

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**DETERMINATION OF THE PLASTIC YIELD CONDITION AS A VARIATIONAL PROBLEM**

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1. *Introduction.*—Dr. G. R. Irwin has suggested that plastic flow occurs in such a way as to minimize a certain energy integral taken over the region of flow. This appears to us to be a particularly significant idea, and we have accordingly attempted to derive certain of its consequences in the following communication. In so doing, we have based our work on the *deviation tensors* of stress and strain rate, in view of the generally recognized fact that plastic phenomena are more or less independent of the effects of hydrostatic pressure. Thus, if $u_\alpha$ denote the components of velocity and $\sigma_{\alpha\beta}$ the ordinary stress components, we have

\[
\varepsilon_{\alpha\beta} = \frac{1}{2} (u_{\alpha, \beta} + u_{\beta, \alpha}); \quad \eta_{\alpha\beta} = \varepsilon_{\alpha\beta} - \frac{\varepsilon_{ij}}{3} \delta_{\alpha\beta}; \quad s_{\alpha\beta} = \sigma_{\alpha\beta} - \frac{\sigma_{ij}}{3} \delta_{\alpha\beta} \quad (1)
\]

for the components of the strain-rate tensor $\varepsilon$, its deviation $\eta$, and the stress-deviation tensor $s$. Actually, the quantities $u_\alpha$ can also be interpreted as displacements in the following discussion, in which case the tensor $\varepsilon$ is the ordinary strain tensor.

Now in the elastic domain the quantity $s_{\alpha\beta} \delta_{\alpha\beta}$ represents, apart from a constant factor depending on the elastic moduli of the material, the distortion energy per unit volume, i.e., the energy per unit volume stored in the body as the result of change in shape as distinct from change in volume. It would be natural to consider the distortion energy in the plastic regime and to deduce the effect of minimiz-
ing the integral of this energy as one of the characteristic conditions governing the plastic flow. However, it is obvious that the expression $s_{a\beta}q_{a\beta}$, derived from purely elastic theory, may not represent, to within a constant factor of proportionality, the plastic distortion energy, except perhaps as a first approximation, since, in the plastic, as distinct from the elastic, case, we are concerned with an irreversible process involving the phenomenon of permanent set, as well as other factors which it may be necessary to take into account in any consideration of energy content.

In the following article we have considered the effect on the plastic flow of the requirement that the integral of an arbitrary differentiable (invariant) function $\psi(s, \eta)$ of the components of the deviation tensors $s$ and $\eta$ have a stationary value over the plastic domain. Strong differential conditions for the stationary character of this integral are derived in section 2 of this article by the method of virtual displacements. Now, when these conditions are combined with the usual dynamical conditions of equilibrium and the usual relation expressing the incompressibility of the medium, we have an overdetermined system. In section 4 we have shown how this overdetermination can be removed, and, in so doing, we are led to an extended system of differential equations for the function $\psi$; this system remains valid for the case of compressible material (sec. 3). Treating this extended system on the basis of a consideration involving canonical co-ordinates, it is shown in section 6 that $\psi$ can, quite generally, be a function at most of the trace $\eta_{tt}$ of the tensor $\eta$. But $\eta_{tt} = 0$, from the definition of the deviation tensor $\eta$, and hence we arrive at the conclusion that $\psi = \text{Const}$. In other words, $\psi$ must reduce to a constant as a function of the components of the $\eta$ tensor. This result is of the nature of a yield condition, as emphasized in section 7.

If, in particular, we assume that the tensors $s$ and $\eta$ are proportional and if we take the function $\psi$ to be given by the expression $s_{a\beta}q_{a\beta}$, we arrive at the quadratic yield condition and stress-strain relations used by von Mises. This same yield condition is also obtained by the selection $\psi = s_{a\beta}q_{a\beta}$ and the most general relation between the tensors $s$ and $\eta$, considered in section 8.

According to the above result, the volume integral of the function $\psi$ must remain unchanged by virtual displacements. This situation, which differs from that in the classical theory of elasticity, where the integral of potential energy attains an absolute minimum in a strong sense, must be viewed as a characteristic property of the plastic flow. That this result is not trivial in spite of its simplicity is evident from the analysis needed for its demonstration, as well as from the fact that it depends essentially on the imposed requirement of nonoverdetermination of the differential relations and on the assumption that the stress deviation $s$ is an invariant of the deviation tensor $\eta$. Now this latter condition is not met in the type of relation assumed between the tensors $s$ and $\eta$ of certain theories of plasticity, e.g., the Prandtl-Reuss theory. Since the stress-strain relationship is closely connected with the nature of the function $\psi$ to be used in this variational procedure, an investigation, based on physical considerations, of some of the most likely choices of the function $\psi$ may throw some light on these matters.

2. Differential Conditions.—Consider a medium under the action of surface forces, and denote by $u_{\alpha}$ the components of the resulting displacement or velocity of the material particles. If the surface forces act only on a part $\Sigma_{1}$ of the boundary, it is assumed that over the remaining part $\Sigma_{2}$ of the boundary the values of the $u_{\alpha}$
are assigned. Now consider a change $\delta u_a$ in the quantities $u_a$, with $\delta u_a$ arbitrary, subject to conditions of continuity and differentiability and to the above boundary conditions, i.e., $\delta u_a = 0$ on the part $\Sigma_1$ of the boundary where the $u_a$ are assigned. Such a displacement $\delta u$ is called a virtual displacement.

For simplicity we suppose the medium under discussion to be entirely in the plastic regime. We now impose the condition that

$$\delta \int_V \psi(s, \eta) dV = 0$$

under the virtual displacement $\delta u$, where the integration is over the entire volume $V$ of the plastic body. In treating condition (2) above, it will be assumed that $s$ is a tensor invariant of the tensor $\eta$, i.e., $s_{\alpha\beta} = s_{\alpha\beta}(\eta)$. The exact form of this relationship will be considered later, as it will not be needed in the immediate discussion.

Now we have

$$\delta \eta_{\alpha\beta} = \frac{1}{2} (\delta u_{\alpha, \rho} + \delta u_{\rho, \alpha}) - \frac{1}{3} \delta u_{\alpha, \rho} \delta_{\alpha\rho} = \frac{1}{2} \left( \frac{\partial \delta u_{\alpha}}{\partial x^\rho} + \frac{\partial \delta u_{\rho}}{\partial x^\alpha} \right) - \frac{1}{3} \frac{\partial \delta u_{\alpha}}{\partial x^\rho} \delta_{\alpha\rho},$$

where the differentiation is with respect to rectangular co-ordinates $x$. Hence it is easily seen that condition (2) above can be written in the form

$$\int_V \frac{\partial \psi}{\partial \eta_{\alpha\beta}} \frac{\partial \delta u_{\alpha}}{\partial x^\rho} dV - \frac{1}{3} \int_V \frac{\partial \psi}{\partial \eta_{\alpha\rho}} \frac{\partial \delta u_{\alpha}}{\partial x^\rho} dV = 0,$$

where the function $\psi$ in this equation is that which results from the original function $\psi(s, \eta)$ by the above-mentioned substitution $s = s(\eta)$. But equation (3) can also be written

$$\left\{ \int_V \left( \frac{\partial \psi}{\partial \eta_{\alpha\rho}} \delta u_{\rho} \right) dV - \int_V \left( \frac{\partial \psi}{\partial \eta_{\alpha\rho}} \right) \delta u_{\rho} dV \right\} - \frac{1}{3} \int_V \left( \frac{\partial \psi}{\partial \eta_{\alpha\rho}} \delta u_{\rho} \right) dV + \frac{1}{3} \int_V \left( \frac{\partial \psi}{\partial \eta_{\alpha\rho}} \delta u_{\rho} dV = 0 \right).$$

Now the first and third integrals in this equation can be expressed as surface integrals by the application of Green's theorem. We thus obtain

$$\int_S \left[ \left( \frac{\partial \psi}{\partial \eta_{\alpha\rho}} \right) - \frac{1}{3} \left( \frac{\partial \psi}{\partial \eta_{\alpha\rho}} \right) \delta u_{\rho} dV - \int_S \left( \frac{\partial \psi}{\partial \eta_{\alpha\rho}} \nu_{\rho} - \frac{1}{3} \frac{\partial \psi}{\partial \eta_{\alpha\rho}} \nu_{\rho} \right) \delta u_{\rho} dS = 0,$$

where $S$ denotes the boundary of the body and $\nu$ is the outward unit normal to $S$. Actually, the surface integral can be restricted to the part $\Sigma_1$ of the boundary, since the quantities $\delta u_{\rho} = 0$ over $\Sigma_2$ by hypothesis. In view of the arbitrary character of the virtual displacement $\delta u$, the above equation is now seen to decompose into the two following differential conditions, namely,

$$\frac{\partial \psi}{\partial \eta_{\alpha\rho}} \nu_{\rho} - \frac{1}{3} \frac{\partial \psi}{\partial \eta_{\alpha\rho}} \nu_{\rho} = 0, \quad \text{over } \Sigma_1.$$
3. Equations of Flow.—It is generally considered admissible to replace the
dynamical equations of motion by the equilibrium conditions $\sigma_{\alpha\beta\gamma} = 0$ in treating
the plasticity problem. Using these conditions and differentiating the last set of
relations (1) we find $s_{\alpha\beta\gamma} = p_{\alpha\gamma}$ where we have put $p = -\sigma_{\alpha\gamma}/3$ for brevity. One
also usually assumes that the plastic material is incompressible, which is expressed
by the relation $u_{\alpha\alpha} = 0$. Combining these conditions and making use of the as-
sumption of section 1 that the tensor $s$ is an invariant of the $\eta$ tensor, we can write

$$\frac{\partial s_{\alpha\beta}}{\partial \eta_{\alpha\beta}} = p_{\alpha\gamma}; \quad u_{\alpha\alpha} = 0.$$  

(7)

These are four relations for the determination of the four quantities $u_{\alpha\gamma}$ and $p_{\alpha\gamma}$. To
take account of compressibility, one sometimes replaces the last equation in (7)
by the corresponding equation derived from ordinary elasticity theory. This leads
to the equations

$$\frac{\partial s_{\alpha\beta}}{\partial \eta_{\alpha\beta}} = p_{\alpha\gamma}; \quad p = -\frac{E}{3(1-2\nu)}u_{\alpha\alpha},$$  

(8)

where $E$ is Young’s modulus and $\nu$ is Poisson’s ratio for the material. Again, we
have four unknown quantities $u_{\alpha\gamma}$ and $p_{\alpha\gamma}$ and four equations (8) for the determination
of these quantities. When the quantities $u_{\alpha\gamma}$ and $p_{\alpha\gamma}$ are found from either (7) or (8),
we can determine the $e_{\alpha\beta}$ and the $\eta_{\alpha\beta}$ from the first two sets of relations (1). Also,
the components $s_{\alpha\beta}$ can be determined from the relations $s = s(\eta)$, and, finally,
the stress components $\sigma_{\alpha\beta}$ can be obtained from the last set of equations (1).

4. Prevention of Overdetermination.—When we add (7) or (8) to the conditions
(5) and (6), we shall in general have an overdetermined system. To prevent this
situation we impose conditions which are necessary and sufficient for the nonexistence
of such overdetermination. Since equations (7) or (8) are already sufficient for the
determination of the quantities $u_{\alpha\gamma}$ and $p_{\alpha\gamma}$ under the assumption that the functions
$s_{\alpha\beta}(\eta)$ are known, this means that (5) and (6) must add no new conditions to the
conditions given by (7) or (8). In particular, (5) and (6) can put no additional
algebraic conditions on the quantities $\eta_{\alpha\beta\gamma}$ over those imposed by (7) or (8). We
shall now determine the precise consequences of this requirement.

The quantities $\eta_{\alpha\beta\gamma}$ satisfy the relations $\eta_{\alpha\alpha\beta} = 0$ from the definition of $\eta$ as a
deviation tensor, and it is obvious that there will be, in general, no additional alge-
braic conditions on the components of this tensor. Now (7) or (8) can put no additional
algebraic conditions on the quantities $\eta_{\alpha\beta\gamma}$ since the first set of equa-
tions in (7) or (8) can always be satisfied by a proper choice of the $p_{\alpha\gamma}$ for any selec-
tion of the $\eta_{\alpha\beta\gamma}$ and the remaining equation in (7) or (8) does not involve these
quantities. Hence it follows from the above discussion that equations (5) must be
satisfied whenever $\eta_{\alpha\alpha\beta} = 0$. But we can write

$$\left(\frac{\partial \eta_{\alpha\beta}}{\partial \eta_{\gamma\delta}}\right)_{\epsilon} = \frac{\partial^2 \psi}{\partial \eta_{\alpha\beta}\partial \eta_{\gamma\delta}} \eta_{\alpha\beta\gamma}\delta_{\epsilon\delta},$$  

$$\left(\frac{\partial \psi}{\partial \eta_{\epsilon\delta}}\right)_{\alpha\beta\gamma} = \frac{\partial^2 \psi}{\partial \eta_{\epsilon\delta}\partial \eta_{\alpha\beta}} \delta_{\epsilon\delta} \eta_{\alpha\beta\gamma}.\, $$

(5)
Hence we have
\[
\left[ \frac{\partial^2 \psi}{\partial \eta_{\alpha\rho} \partial \eta_{\beta\rho}} - \frac{1}{3} \frac{\partial^2 \psi}{\partial \eta_{\gamma\eta} \partial \eta_{\alpha\beta}} \delta_{\alpha\rho} \right] \eta_{\alpha\beta, \rho} = 0,
\]
whenever \( \eta_{\alpha\alpha, \rho} = 0 \). From symmetry considerations we are led to express this condition as follows:
\[
\left[ \frac{\partial^2 \psi}{\partial \eta_{\alpha\rho} \partial \eta_{\beta\rho}} - \frac{1}{3} \frac{\partial^2 \psi}{\partial \eta_{\gamma\eta} \partial \eta_{\alpha\beta}} \delta_{\alpha\rho} \right] \eta_{\alpha\beta, \rho} = 0,
\]
whenever \( \eta_{\alpha\alpha, \rho} = 0 \).

For brevity let us denote the above symmetrical bracketed expression by \( < \sigma \rho \alpha \beta > \), so that our condition becomes
\[
< \sigma \rho \alpha \beta > \eta_{\alpha\beta, \rho} = 0, \quad \text{whenever} \quad \eta_{\alpha\alpha, \rho} = 0.
\]

But we can also write
\[
< \sigma \rho \alpha \beta > = A_{\sigma \rho} \delta_{\alpha \beta}, \quad \text{whenever} \quad \eta_{\alpha\alpha, \rho} = 0,
\]
with arbitrary quantities \( A_{\sigma \rho} \). Now three of these quantities \( \eta \), e.g., the three components \( \eta_{11,1} \), \( \eta_{11,2} \), and \( \eta_{11,3} \), can be selected as dependent, in that they can be considered to be determined by the relations \( \eta_{\alpha\alpha, \rho} = 0 \), after all other components \( \eta_{\alpha\beta, \gamma} \) have been selected arbitrarily. Choose the \( A \)'s in the above relations (9) by the condition \( A_{\sigma \rho} = < \sigma \rho 11 > \). Then the coefficients of all dependent quantities \( \eta_{\alpha\beta, \rho} \) vanish in the relations (9). The coefficients of the remaining quantities \( \eta_{\alpha\beta, \rho} \) must therefore likewise vanish, owing to the arbitrary character of these quantities. Hence we have
\[
< \sigma \rho \alpha \beta > = A_{\sigma \rho} \delta_{\alpha \beta}.
\]

But from symmetry \( < \sigma \rho \alpha \beta > = < \alpha \beta \sigma \rho > \). Hence, from (10), it follows that
\[
A_{\sigma \rho} \delta_{\alpha \beta} = A_{\alpha \beta} \delta_{\sigma \rho}.
\]
Putting \( \alpha = \beta \) in these relations and summing on the repeated index, we now obtain
\[
A_{\sigma \rho} = \frac{1}{3} A_{\alpha \beta} \delta_{\sigma \rho} = B \delta_{\sigma \rho}.
\]

Hence (10) becomes
\[
< \sigma \rho \alpha \beta > = B \delta_{\sigma \rho} \delta_{\alpha \beta}.
\]

In the treatment of relations (11), we have found it expedient to avail ourselves of the simplification afforded by the use of canonical co-ordinates. In the following section we develop the necessary formulas of differentiation relative to these co-ordinates.

5. Canonical Co-ordinates.—Consider the set of three equations
\[
(\eta_{\alpha\beta} - \tau_{\delta_{\alpha\beta}}) h_{i}^{\beta} = 0, \quad i \text{ not summed}.
\]

The quantities \( \tau_{i} \) in these equations are the principal values of the tensor \( \eta \) and are given as solutions of the equation formed by equating to zero the determinant of
the coefficients of the \( h_i^\beta \). Since this determinant is of the third order, there will be three quantities \( \tau_i \) which in general will have distinct values; these values can be shown to be real. **We assume this general case in the following discussion.** Then the three vectors \( h_i \) which are determined as solutions of (12) will be unique to within algebraic sign and can be shown to be mutually orthogonal, i.e.,

\[
h_i^a h_i^\beta = \delta^{a\beta}, \quad \text{or} \quad h_i^a h_j^a = \delta_{ij}.
\]  

(13)

We may suppose the designation of the vectors \( h_i \) to be such that the vector triad \( h_1, h_2, h_3 \) has the same orientation as the co-ordinate axes \( x^1, x^2, x^3 \). The rectangular co-ordinate system \( y \) having its origin at any point \( P \) and such that its \( y^1, y^2, y^3 \) axes coincide with the directions \( h_1, h_2, h_3 \) at \( P \) will be related to the underlying \( x \) system by a proper orthogonal transformation. Such a co-ordinate system \( y \) will be called **canonical.** An obvious property of the canonical system is that relative to this system and at its origin we have \( \eta_{a\beta} = \tau_i \delta_{a\beta} \). In fact, this relation follows immediately by considering equation (12) relative to canonical co-ordinates, since \( h_i^\beta = \delta_i^\beta \) at the origin of the canonical system.

Multiplying (12) by \( h_j^\alpha \) and summing on the repeated index \( \alpha \), we see that these equations can be written in the form

\[
\tau_{i\beta} = \eta_{a\beta} h_i^a h_j^\beta.
\]  

(14)

We now differentiate relations (13) and (14) partially with respect to the \( \eta_{a\rho} \). In carrying out this differentiation, it is advantageous from the formal standpoint to suppose that the quantities \( \eta_{a\rho} \) are completely independent and to take account of such properties as the symmetry of the components \( \eta_{a\rho} \) and the fact that the actual tensor \( \eta \) has a vanishing trace after all differentiations have been completed. Hence, from the second set of relations (13), we obtain

\[
\frac{\partial h_i^a}{\partial \eta_{a\rho}} h_j^\alpha + h_i^a \frac{\partial h_j^\alpha}{\partial \eta_{a\rho}} = 0.
\]  

(15)

Similarly, by differentiation of (14), we are led to the relations

\[
\frac{\partial \tau_{i\beta}}{\partial \eta_{a\rho}} \delta_{ij} = \delta_{a\beta} \delta_{a\rho} h_i^a h_j^\beta + \eta_{a\beta} \frac{\partial h_i^a}{\partial \eta_{a\rho}} h_j^\beta + h_i^a \frac{\partial h_j^\beta}{\partial \eta_{a\rho}}
\]

\[
= h_i^\alpha h_j^\alpha + \tau_{ij} h_j^\alpha \frac{\partial h_j^\alpha}{\partial \eta_{a\rho}} + \tau_{ij} \frac{\partial h_j^\beta}{\partial \eta_{a\rho}},
\]

when use is made of (12). But on account of (15) these latter relations become

\[
\frac{\partial \tau_{i\beta}}{\partial \eta_{a\rho}} \delta_{ij} = h_i^\alpha h_j^\beta + (\tau_{ij} - \tau_{ij}) h_i^a \frac{\partial h_j^\alpha}{\partial \eta_{a\rho}}.
\]

Hence, according as \( i = j \) or \( i \neq j \), we now have

\[
\frac{\partial \tau_{i\beta}}{\partial \eta_{a\rho}} = h_i^\alpha h_j^\beta,
\]  

(16)

\[
(\tau_{ij} - \tau_{ij}) h_i^a \frac{\partial h_j^\alpha}{\partial \eta_{a\rho}} + h_i^\alpha h_j^\beta = 0, \quad i \neq j.
\]  

(17)
The above relations (15), (16), and (17) are valid in arbitrary rectangular coordinate systems. Transforming to canonical co-ordinates and evaluating at the origin of the canonical system, we obtain

\[ \frac{\partial h_i^i}{\partial \eta_{\sigma \rho}} + \frac{\partial h_i^j}{\partial \eta_{\sigma \rho}} = 0, \]  
(18)

\[ \frac{\partial \tau_i}{\partial \eta_{\sigma \rho}} = \delta_i^\sigma \delta_i^\rho, \]  
(19)

\[ \frac{\partial h_i^i}{\partial \eta_{\sigma \rho}} = \frac{-\delta_i^\sigma \delta_i^\rho}{\tau_i - \tau_j}, \quad i \neq j. \]  
(20)

Equations (18) have the simple interpretation that the derivatives of the $h_i^j$ are skew-symmetric in the indices $i$ and $j$. Hence, in particular, if these indices have equal values, the derivative must vanish.

6. Condition on $\psi$.—The scalar invariant $\psi(\eta)$ of the $\eta$ tensor can be regarded as a function $\psi(\tau)$ of the three principal values $\tau_1, \tau_2, \tau_3$ of this tensor. This is seen immediately by transforming the invariant $\psi(\eta)$ to canonical co-ordinates and then using the fact that $\eta_{\alpha \beta} = \tau_\alpha \delta_{\alpha \beta}$ at the origin of the canonical system. Hence, by differentiation, we have

\[ \frac{\partial \psi}{\partial \eta_{\sigma \rho}} = \frac{\partial \psi}{\partial \tau_i} \frac{\partial \tau_i}{\partial \eta_{\sigma \rho}} = \frac{\partial \psi}{\partial \tau_i} h_i^\sigma h_i^\rho, \]  
(21)

when use is made of (16). There is a summation on the index $i$, of course, in these relations. Again differentiating (21), we find that

\[ \frac{\partial^2 \psi}{\partial \eta_{\sigma \rho} \partial \eta_{\alpha \beta}} = \frac{\partial^2 \psi}{\partial \tau_i \partial \tau_j} h_i^\sigma h_i^\rho h_j^\sigma h_j^\rho + \frac{\partial \psi}{\partial \tau_i} \frac{\partial h_i^\sigma}{\partial \eta_{\alpha \beta}} h_i^\rho + \frac{\partial \psi}{\partial \tau_i} \frac{\partial h_i^\rho}{\partial \eta_{\alpha \beta}} h_i^\sigma. \]

At the origin of canonical co-ordinates these equations become

\[ \frac{\partial^2 \psi}{\partial \eta_{\sigma \rho} \partial \eta_{\alpha \beta}} = \frac{\partial^2 \psi}{\partial \tau_i \partial \tau_j} \delta_{\alpha \beta} \delta_{\sigma \rho} + \left( \frac{\partial \psi}{\partial \tau_\rho} - \frac{\partial \psi}{\partial \tau_\rho} \right) \frac{\partial h_i^\sigma}{\partial \eta_{\alpha \beta}} \]  

as we see immediately when we make the substitution $h_i^\sigma = \delta_i^\sigma$ and use relations (18). Hence, in particular, we can write

\[ \frac{\partial^2 \psi}{\partial \eta_{\sigma \rho} \partial \eta_{\alpha \beta}} = \left( \frac{\partial \psi}{\partial \tau_\rho} - \frac{\partial \psi}{\partial \tau_\rho} \right) \frac{\partial h_i^\sigma}{\partial \eta_{\alpha \beta}}, \quad \alpha \neq \beta; \quad \sigma \neq \rho, \]  
(22)

at the origin of the canonical system.

Let us now consider the invariant relations (11) in canonical co-ordinates, and let us then evaluate these relations at the origin of this system. Choosing $\sigma \neq \rho$ and $\alpha \neq \beta$, we then have

\[ < \sigma \rho \alpha \beta > = \frac{\partial^2 \psi}{\partial \eta_{\sigma \rho} \partial \eta_{\alpha \beta}} = 0, \quad \alpha \neq \beta; \quad \sigma \neq \rho. \]  
(23)
Comparison of (22) and (23) now yields
\[
\left( \frac{\partial \psi}{\partial \tau_\rho} - \frac{\partial \psi}{\partial \tau_\sigma} \right) \frac{\partial h_\rho^\sigma}{\partial \eta_{\alpha\beta}} = 0, \quad \alpha \neq \beta; \quad \sigma \neq \rho,
\]
or
\[
\left( \frac{\partial \psi}{\partial \tau_\rho} - \frac{\partial \psi}{\partial \tau_\sigma} \right) \frac{\delta_\sigma^\alpha \delta_\rho^\beta}{\tau_\sigma - \tau_\rho} = 0, \quad \alpha \neq \beta; \quad \sigma \neq \rho,
\]
on account of (20). In these relations there is, of course, no summation on repeated indices, and \( \tau_\sigma \neq \tau_\rho \), since \( \sigma \neq \rho \) and the \( \tau \)'s are distinct by hypothesis. Hence, taking \( \alpha = \sigma \) and \( \beta = \rho \) without summation, it follows that \( \partial \psi / \partial \tau_\alpha = \partial \psi / \partial \tau_\beta \) for \( \alpha \neq \beta \). In other words, we must have
\[
\frac{\partial \psi}{\partial \tau_1} = \frac{\partial \psi}{\partial \tau_2} = \frac{\partial \psi}{\partial \tau_3}, \tag{24}
\]
While these relations have been established at the origin of canonical co-ordinates by the above process, it is clear that they actually hold without regard to the coordinate system, on account of their invariant character.

From (21) and (24), we now have
\[
\frac{\partial \psi}{\partial \eta_{\alpha\rho}} = \frac{\partial \psi}{\partial \tau_1} \sum_i h_i^\rho h_i^\sigma = \frac{\partial \psi}{\partial \tau_1} \delta_\sigma^\rho, \tag{25}
\]
or
\[
\frac{\partial \psi}{\partial \eta_{\alpha\alpha}} = \frac{\partial \psi}{\partial \tau_1}, \quad \alpha \text{ not summed},
\]
\[
\frac{\partial \psi}{\partial \eta_{\alpha\beta}} = 0, \quad \alpha \neq \beta. \tag{26}
\]
It is immediately seen from (25) and (26) that the invariant \( \psi \) is a function of the trace \( \Sigma \eta_{\alpha\alpha} \) of the \( \eta \) tensor. A simple formal demonstration of this fact can be given as follows: Observing that the function \( \psi(\eta) \) does not depend on the components \( \eta_{\alpha\beta} \) for \( \alpha \neq \beta \) on account of (26), let us make the substitution
\[
\eta_{11} = \tilde{\eta}_{11}; \quad \eta_{22} = \tilde{\eta}_{22}; \quad \eta_{33} = \tilde{\eta}_{33} - \eta_{11} - \eta_{22}. \tag{27}
\]
Suppose that \( \psi(\eta) \) becomes \( \tilde{\psi}(\tilde{\eta}) \) as the result of this substitution. Then, taking account of (25), we have
\[
\frac{\partial \tilde{\psi}}{\partial \tilde{\eta}_{11}} = \frac{\partial \psi}{\partial \eta_{11}} - \frac{\partial \psi}{\partial \eta_{33}} = 0; \quad \frac{\partial \tilde{\psi}}{\partial \tilde{\eta}_{22}} = \frac{\partial \psi}{\partial \eta_{22}} - \frac{\partial \psi}{\partial \eta_{33}} = 0.
\]
Hence \( \tilde{\psi} \) depends only on the single quantity \( \eta_{33} \), which, from (27), is seen to be the trace of the \( \eta \) tensor.

Denoting by \( \theta \) the trace of the tensor \( \eta \), the above result can be expressed by saying that \( \psi \) is a function of \( \theta \) alone. Hence
\[
\frac{\partial \psi}{\partial \eta_{\rho\rho}} = \psi'(\theta) \delta_\rho^\rho; \quad \frac{\partial \psi}{\partial \eta_{\rho\rho}} = 3 \psi'(\theta),
\]
in which the prime denotes differentiation with respect to \( \theta \). Also,
\[
\frac{\partial \psi}{\partial \eta_{ss}} = \psi'(\theta)\eta_{s} \quad ; \quad \frac{\partial \psi}{\partial \eta_{pp}} = 3\psi'(\theta)\eta_{s}.
\]

Hence conditions (5) are satisfied by the function \( \psi = \psi(\theta) \). Turning now to the boundary relations (6), which were not used in the determination of the above form of the function \( \psi \), we have
\[
\frac{\partial \psi}{\partial \eta_{sp}} \eta_{s} = \psi'(\theta)\eta_{s} ; \quad \frac{\partial \psi}{\partial \eta_{pp}} \eta_{s} = 3\psi'(\theta)\eta_{s}.
\]

It follows that the function \( \psi = \psi(\theta) \) automatically satisfies conditions (6). In other words, \( \psi(\theta) \) represents the most general form of the function \( \psi \) satisfying the variational condition (2) under the above hypotheses.

7. The Yield Condition.—If we now take cognizance of the fact that \( \eta \) is a devi- ation tensor, so that its trace \( \theta = 0 \), it follows that \( \psi = \text{const} \). In other words, the function \( \psi(s, \eta) \) used originally in our variational problem must actually reduce to a constant as a function of the components \( \eta_{ab} \). This result is of the nature of a yield condition. For illustration, suppose that \( \psi = s_{ab}\eta_{ab} \) and that the relation between the tensors \( s \) and \( \eta \) is given by \( s_{ab} = \phi\eta_{ab} \), where the proportionality factor \( \phi \) is here to be thought of as a scalar invariant of the tensor \( \eta \). Then, from the above result, we have the von Mises or quadratic yield condition \( s_{ab} \eta_{ab} = 2k^2 \), where \( k \) is a constant depending on the material properties of the medium. Hence \( \phi^2\eta_{ab}\eta_{ab} = 2k^2 \), or \( \phi = \sqrt{2 k/\sqrt{\eta_{ij}\eta_{ij}}} \), and hence
\[
s_{ab} = \frac{\sqrt{2 k\eta_{ab}}}{\sqrt{\eta_{ij}\eta_{ij}}} \quad . \quad (28)
\]

This relation between the tensor \( s \) and \( \eta \) is precisely that used by von Mises in his theory of plasticity.

8. General Relation between the \( s \) and \( \eta \) Tensors.—It has been shown by Erickson and Rivlin \( ^4 \) that if the tensor \( s \) in an invariant of the tensor \( \eta \), we must have
\[
s_{ab} = F\delta_{ab} + J\eta_{ab} + K\eta_{aa}\eta_{bb}
\]
for the three-dimensional case under consideration, where the quantities \( F \), \( J \), and \( K \) are scalar invariants of the \( \eta \) tensor with respect to orthogonal transformations. Imposing the condition that the trace of the tensors \( s \) and \( \eta \) must vanish, it follows that \( 3F = -K\eta_{ij}\eta_{ij} \). Using this relation to eliminate the scalar \( F \) from the above equations, we thus obtain
\[
s_{ab} = J\eta_{ab} + K\eta_{aa}\eta_{bb} - 1/3K(\eta_{ij}\eta_{ij})\delta_{ab} \quad . \quad (29)
\]
Selection of the function \( \psi(s, \eta) \) will now lead to a relation between the scalars \( J \) and \( K \). Thus, taking \( \psi = s_{ab}\eta_{ab} \), we have
\[
\eta_{ij}\eta_{ij}F^2 + 2JK\eta_{aa}\eta_{bb}\eta_{ab} + K^2[\eta_{aa}\eta_{bb}\eta_{ab} - 1/3(\eta_{ij}\eta_{ij})^2] = \text{const.}
\]
Having recourse to canonical co-ordinates, we see that we can now write
These relations define the scalar invariants \( \tau_1 \), \( \tau_2 \), \( \tau_3 \), and \( \tau_4 \) in terms of the components \( \eta_{ab} \) and also give their values in terms of the principal values \( \tau_i \) of the \( \eta \) tensor. Now we can easily establish the following general identity:

\[
\tau_4 = \tau_{(2)}^2 + 4\tau_1\tau_2\tau_3\tau_1 - \frac{1}{2} \left[ \tau_{(2)} - \tau_{(3)}^2 \right]^2.
\]

But \( \tau_1 = 0 \) from the special character of \( \eta \) as a deviation tensor in our problem, and hence the above identity reduces to \( \tau_4 = \tau_{(2)}^2/2 \). Using this result and putting \( L = K/J \), the above relation between \( J \) and \( K \) can now be written as

\[
\left[ \tau_{(2)} + 2\tau_{(3)}L + \frac{1}{6} \tau_{(2)}^2L^2 \right] J^2 = 2k^2,
\]

where \( k \) is a material constant. We thus have

\[
J = \frac{\sqrt{2k}}{\sqrt{\tau_{(2)} + 2\tau_{(3)}L + \frac{1}{6} \tau_{(2)}^2L^2}}, \quad K = \frac{\sqrt{2kL}}{\sqrt{\tau_{(2)} + 2\tau_{(3)}L + \frac{1}{6} \tau_{(2)}^2L^2}}.
\]

Hence the stress-strain relation (29) involves, in addition to the usual constant \( k \), an invariant of the material in the form of the scalar function \( L \). For \( L = 0 \), we have \( K = 0 \), and (29) reduces to the usual stress-strain relation (28).

1 We use the summation convention in accordance with which an index which appears more than once in a term is to be summed over the values 1, 2, 3. Several exceptions to this rule occur, but these exceptions should be clear from the context. However, to prevent possible confusion in some of the formulas, or for additional emphasis, the summation sign, \( \Sigma \), has been used in certain instances.

2 Dr. H. Trent has suggested that the plastic region should be considered to be surrounded by an elastic region, since this would more closely conform with the actual situation which occurs when the plastic state is established as the result of applied stresses or displacements over the boundary of the body. We agree, of course, with this point of view. Actually, it might be advisable to consider also the possibility of a departure from the strict linearity of the relationships of the classical elasticity theory, since such linearity does not appear to persist up to the beginning of plastic flow. In neglecting these and other factors which might be mentioned, it is possible that we have been guilty of oversimplification of the problem. However, it is likely that the analytical procedures here employed can be incorporated in the treatment of such modifications of the point of view which we have adopted.

3 The definition of the overdetermined system which is here adopted is the usual one, namely, that overdetermination occurs when the number of equations exceeds the number of dependent or unknown functions. It is evident that in general overdetermination will result in inconsistency in some sense or in incompatibility with the required boundary conditions of the problem. An example of such incompatibility with boundary conditions is found when one attempts to solve the shock-wave problem which arises when an airfoil is placed in a uniform supersonic stream and it is assumed that the flow is everywhere irrotational or isentropic.