INTERDEPENDENCE OF THE YIELD CONDITION AND THE STRESS-STRAIN RELATIONS FOR PLASTIC FLOW

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1. The Fundamental Assumptions.—Denote by $s_{ab}$ the components of the deviation $s$ of the stress tensor $\sigma$. Let $v_a$ be the components of the velocity vector $v$ and $\epsilon_{ab}$ the components of the rate of strain tensor $\epsilon$ derived from the velocity vector. We impose the usual condition that the flow is incompressible, i.e., $\epsilon_{aa} = 0$, which appears to be in agreement with experimental evidence. Hence $\epsilon$ is identical with its deviation. We now make the following fundamental assumptions:

I. The stress deviation $s$ is an invariant of the tensor $\epsilon$ under proper orthogonal coordinate transformations. In other words we have

$$ s_{ab} = \Phi_{ab}(\epsilon_{11}, \epsilon_{12}, \ldots, \epsilon_{33}), $$

where the quantities $\Phi_{ab}$ are the components of a symmetric tensor invariant of the tensor $\epsilon$ relative to the proper orthogonal group. Relations (1) will be called the stress-strain relations. In addition to the components $\epsilon_{ab}$ which appear explicitly in the right-hand members of (1), the quantities $\Phi$ may also involve, subject to their invariant character, point functions, e.g., scalar functions of the co-ordinates, and indeed terms depending, in some sense, on the strain history of the material, which is known to influence present behavior. The restriction to rectangular co-ordinates which is implied by Assumption I is natural and constitutes no loss of generality, since one can readily express relations (1), as well as all following relations, in arbitrary co-ordinates by the ordinary methods of the tensor analysis. Use of the deviation tensor $s$ rather than the actual stress tensor $\sigma$ in (1) has its justification in the experimental fact that plastic flow is influenced but little, if at all, by the effects of hydrostatic pressure; the precise argument which is made in this connection is well known and will not be repeated. Relations (1) include, in particular, the stress-strain relations of the von Mises plasticity theory. They do not strictly contain the Prandtl-Reuss stress-strain relations, since these involve, in addition to the above tensors $s$ and $\epsilon$, a tensor whose components are the time derivatives of the components of the stress deviation. By extending Assumption I to include this latter tensor, we can deduce the dependence of the yield condition on general stress-strain relations of the Prandtl-Reuss type.

II. Relations (1) do not establish a (1, 1) correspondence between the components of the $s$ and $\epsilon$ tensors. There appears to be good experimental evidence for this assumption, which may, in fact, be regarded as expressing the essential distinction between plastic flow and the flow of ordinary fluids.

2. Scalar Invariants.—At the origin of a system of canonical co-ordinates for the tensor $\epsilon$, the components $\epsilon_{ab}$ of this tensor, relative to the canonical system, have the form $\epsilon_{ab} = \lambda_a \delta_{ab}$, in which there is evidently no summation on the index $a$. The $\lambda_a$ are the so-called principal values or principal invariants of $\epsilon$ and are given as the three solutions $\lambda$, which are necessarily real, of the determinantal equation

$$ |\epsilon_{ab} - \lambda \delta_{ab}| = -\lambda^3 + \frac{1}{2} \epsilon \lambda + \frac{1}{3} \lambda = 0, $$

(2)
where $\xi$ and $\zeta$ are the scalar invariants defined by $\epsilon_{\alpha\beta}\epsilon_{\alpha\beta}$ and $\epsilon_{\alpha\sigma}\epsilon_{\beta\rho}\epsilon_{\alpha\beta}$, respectively. In writing equation (2) we have availed ourselves of the simplicity afforded by the fact that $\epsilon_{\alpha\alpha} = 0$ (see sec. 1). \textit{Hence the principal invariants $\lambda_1$, $\lambda_2$, and $\lambda_3$ are algebraic functions of the scalar invariants $\xi$ and $\zeta$.}

By the process of transforming the component of any scalar invariant $F$ to canonical co-ordinates and evaluating at the origin of this system, it is seen that $F$ can be expressed as a function of the principal invariants $\lambda$. Hence $F$ is a function of the invariants $\xi$ and $\zeta$ by the above italicized statement.

Define the invariant $I_n$ as the combination $\lambda_1^n + \lambda_2^n + \lambda_3^n$ for $n \geq 2$. Now by the use of canonical co-ordinates it is readily seen that we have relations of the type

$$
\begin{align*}
I_2 &= \xi = \epsilon_{\alpha\beta}\epsilon_{\alpha\beta}; \\
I_3 &= \zeta = \epsilon_{\alpha\sigma}\epsilon_{\beta\rho}\epsilon_{\alpha\beta}; \\
I_4 &= \epsilon_{\alpha\sigma}\epsilon_{\beta\tau}\epsilon_{\gamma\rho}\epsilon_{\alpha\beta}. \\
\end{align*}
$$

Each invariant $I_n$ is a function of the two invariants $\xi$ and $\zeta$ by the above result. Calculation of these functions for the invariants $I_4$, $I_5$, and $I_6$, which will enter in the following discussion, leads to the relations

$$
\begin{align*}
I_4 &= \frac{\xi^2}{2}; \\
I_5 &= \frac{5\xi^2}{6}; \\
I_6 &= \frac{\xi^3}{4} + \frac{\zeta^2}{3}.
\end{align*}
$$

3. \textit{General Form of the Stress-Strain Relations.}—It can be shown that any tensor $s$, symmetric and of the second order, which is an invariant of the tensor $\epsilon$ under the group of proper orthogonal transformations, is expressible by relations of the form

$$
\begin{align*}
s_{\alpha\beta} &= F\delta_{\alpha\beta} + G\epsilon_{\alpha\beta} + H\epsilon_{\alpha\alpha}\epsilon_{\beta\beta}, \\
\end{align*}
$$

where $F$, $G$, and $H$ are scalar invariants of the tensor $\epsilon$. Hence a relation (5) exists between the stress deviation $s$ and the rate of strain tensor $\epsilon$, by Assumption I. Now $s_{\alpha\alpha} = 0$ and $\epsilon_{\alpha\alpha} = 0$. We can therefore eliminate the invariant $F$ from (5) to obtain

$$
\begin{align*}
s_{\alpha\beta} &= G\epsilon_{\alpha\beta} + H(\epsilon_{\alpha\alpha}\epsilon_{\beta\beta} - \frac{1}{3}\epsilon_{ij}\epsilon_{ij}\delta_{\alpha\beta}). \\
\end{align*}
$$

But the scalar invariants $G$ and $H$ are functions of the invariants $\xi$ and $\zeta$ by the result in section 2. \textit{Hence the most general relationship between the tensors $s$ and $\epsilon$ is given by (6), in which $G$ and $H$ are functions of the scalars $\xi$ and $\zeta$.}

4. \textit{Proportionality of the $s$ and $\epsilon$ Tensors.}—Assume that relations (6) are satisfied with $H = 0$. Then

$$
\begin{align*}
s_{\alpha\beta} &= G\epsilon_{\alpha\beta},
\end{align*}
$$

i.e., the tensors $s$ and $\epsilon$ are proportional. We first consider the special case of (7) for which the invariant $G$ is a function of $\xi$ alone. We have

$$
\begin{align*}
s_{\alpha\beta}\delta_{\alpha\beta} &= G^2\epsilon_{\alpha\beta}\epsilon_{\alpha\beta} = G^2\xi.
\end{align*}
$$

Now, if $G^2$ $\xi$ is not independent of $\xi$, equation (8) can be solved to express $\xi$ as a uniquely determined function of the quantity $s_{\alpha\beta}\delta_{\alpha\beta}$ by the implicit function theorem. Hence the $\epsilon_{\alpha\beta}$ are determined as functions of the variables $s_{\alpha\beta}$ by (7), and a (1, 1) correspondence between the components of the $s$ and $\epsilon$ tensors is thus established, in contradiction to Assumption II. To avoid this situation we must have
\[ G^2 \xi = \text{const.} \] Hence we can write \( s_{ab} s_{ab} = 2k^2 \), where \( k \) is a constant in the sense that it is independent of the variables \( \xi \) and \( \zeta \) but may depend on other factors associated with the present or past state of the medium (see sec. 1). This gives the value \( G = \sqrt{2k/\sqrt{\xi}} \) for the scalar invariant \( G \) in (7). In other words, we are led to the quadratic yield condition and the stress-strain relations

\[ s_{ab} = \frac{\sqrt{2k}}{\sqrt{\epsilon_{ij} \epsilon_{ij}}} \epsilon_{ab}, \quad (9) \]

used by von Mises in his theory of plasticity.

We now consider the general case of the proportionality relation (7). Put \( \xi_* = s_{ab} s_{ab} \) and \( \zeta_* = s_{ae} s_{eb} s_{ab} \), corresponding to the definition of the scalars \( \xi \) and \( \zeta \) in section 2. Then, from (7), we have

\[ \xi_* = G^2(\xi, \zeta) \xi; \quad \zeta_* = G^3(\xi, \zeta) \zeta. \quad (10) \]

If the functional determinant \( \Delta \) of the right-hand members of equations (10) does not vanish, these equations have a solution \( \xi(\xi_*, \zeta_*), \zeta(\xi_*, \zeta_*) \), and hence \( G \) becomes a function of \( \xi_* \) and \( \zeta_* \). Hence equations (7) admit a solution \( \epsilon_{ab}(s) \) which contradicts Assumption II. We must therefore have \( \Delta = 0 \) to avoid this contradiction.

Equating \( \Delta \) to zero, we find

\[ 2\xi \frac{\partial G}{\partial \xi} + 3\zeta \frac{\partial G}{\partial \zeta} + G = 0. \quad (11) \]

To solve equation (11), we first make the transformation \( \xi = e^{2x}, \zeta = e^{2y} \) of the variables \( \xi, \zeta \) and follow this by the transformation \( x = \alpha, y = \alpha - \beta \). Then (11) becomes \( \partial G/\partial \alpha + G = 0 \), and this equation has the general solution \( G = A(\beta) e^{-\alpha} \), where \( A(\beta) \) is an arbitrary differentiable function of \( \beta \). Returning to the original variables \( \xi, \zeta \), we thus find that the general solution of (11) is given by

\[ G = \frac{E(\sqrt{\epsilon_{ik} \epsilon_{ij} \epsilon_{ij}/\sqrt{\epsilon_{il} \epsilon_{ij}}})}{\sqrt{\epsilon_{il} \epsilon_{ij}}} \quad (12) \]

where \( E \) is an arbitrary differentiable function of the ratio \( \sqrt{\epsilon}/\sqrt{\xi} \).

5. The Yield Condition.—Consider the stress-strain relations (7), with \( G \) given by (12). From (7) we deduce the two equations (10), which can be combined to give \( s_{abc} s_{abc} = \frac{\sqrt{G}}{\sqrt{\epsilon_{ij} \epsilon_{ij}}} \). Hence, replacing the argument \( \sqrt{\epsilon}/\sqrt{\xi} \) of \( E \) by the ratio \( \sqrt{\epsilon}/\sqrt{\xi} \), it follows from (7) and (12) that

\[ s_{ab} = \frac{E(\sqrt{8 s_1 s_2 s_1 s_2}/\sqrt{s_1 s_2 s_1 s_2})}{\sqrt{\epsilon_{ij} \epsilon_{ij}}} \epsilon_{ab}. \quad (13) \]

Now multiply each member of (13) by itself and sum on the repeated indices \( \alpha \) and \( \beta \). We thus obtain the yield condition

\[ s_{ab} s_{ab} = E \left( \frac{\sqrt{8 s_1 s_2 s_1 s_2}}{\sqrt{s_1 s_2 s_1 s_2}} \right)^2. \quad (14) \]

The following result has now been proved: When the tensors \( s \) and \( \epsilon \) are propor-
tional, the yield condition may be taken to have the form (14), in which $E$ is any differentiable function of the ratio $\xi^* / \xi^*$; after selection of the function $E$, the proportionality factor $G$ in (7) will be given by (12). In particular, if we take $E = \sqrt{2}k$, where $k$ is a material constant, we obtain from (14) the von Mises, or quadratic, yield condition, while (7) and (12) give (9) as the associated stress-strain relations.

6. Discussion of the General Case.—Consider the general stress-strain relations (6), in which $G$ and $H$ are functions of the scalar invariants $\xi$ and $\xi$. The formal construction of the equations corresponding to (10) will initially involve homogeneous scalar invariants formed from the components $\epsilon_{i*}$. These homogeneous invariants can be identified with the invariants $\xi, \xi, I_4, I_5$, and $I_6$ by (3), and the invariants $I_4, I_5$, and $I_6$ can then be eliminated by means of (4). We thus obtain

$$
\begin{align*}
\xi^* &= \xi G^2 + 2\xi GH + \frac{1}{6} \xi^2 H^2, \\
\xi^* &= \xi G^2 + \frac{1}{2} \xi^2 G^2 H + \frac{1}{2} \xi^2 GH^2 + \frac{1}{3} \left( \xi^2 - \frac{1}{12} \xi \right) H^3.
\end{align*}
$$

The procedure is now analogous to that employed in section 5. We first observe that the functional determinant $\Delta$ of the right-hand members of (15) must vanish in consequence of Assumption II. This gives a partial differential equation $\Delta = 0$ for the determination of $G$ and $H$ as functions of the variables $\xi$ and $\xi$. Corresponding to any solution $G(\xi, \xi)$ and $H(\xi, \xi)$ of this differential equation, the quantities $\xi^*$ and $\xi^*$ given by (15) will be functionally dependent. The equation expressing this functional dependence will be the yield condition associated with the stress-strain relations (6), in which $G$ and $H$ are the functions obtained as the solution of the above differential equation.

7. Formulation of Results in Terms of Principal Invariants.—Denote by $\lambda_1^*$, $\lambda_2^*$, and $\lambda_3^*$ the principal invariants of the stress deviation $s$, and, for definiteness, let us suppose that these invariants are labeled so that the inequalities $\lambda_1^* \geq \lambda_2^* \geq \lambda_3^*$ are satisfied. In the formulation of results in terms of principal invariants, we shall limit our attention to the proportional case (7), for which we have the simple relationship $\lambda_i^* = G\lambda_i$ ($i = 1, 2, 3$) between the invariants $\lambda^*$ and the principal invariants $\lambda$ of the rate of strain tensor $\epsilon$. Making the customary assumption that $G > 0$, the above inequalities on the $\lambda^*$ imply the inequalities $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Instead of the two variables $\xi$ and $\xi$ used in the above discussion, we can now select two of the three principal invariants $\lambda$. Choose these to be the invariants $\lambda_1$ and $\lambda_3$. The intermediate invariant $\lambda_2$ in the above inequalities is then determined by $\lambda_1$ and $\lambda_3$ on account of the relation $\lambda_1 + \lambda_2 + \lambda_3 = 0$, which expresses the condition that $\epsilon$ is a deviator tensor.

It was seen in section 2 that the component $F$ of any scalar invariant of the $\epsilon$ tensor can be expressed as a function of the principal invariants $\lambda$. Hence $F$ can be expressed as a function of the two invariants $\lambda_1$ and $\lambda_3$ by elimination of the intermediate invariant $\lambda_2$. Hence the coefficient $G$ in (7) can be regarded as a function of the invariants $\lambda_1$ and $\lambda_3$.

We now consider the two relations

$$
\lambda_1^* = G\lambda_1; \quad \lambda_3^* = G\lambda_3.
$$

(16)
From the preceding discussion we can immediately state that Assumption II requires the vanishing of the functional determinant

\[
\begin{vmatrix}
\frac{\partial G}{\partial \lambda_1} \lambda_1 + G & \frac{\partial G}{\partial \lambda_1} \lambda_1 \\
\frac{\partial G}{\partial \lambda_3} \lambda_1 & \frac{\partial G}{\partial \lambda_3} \lambda_3 + G
\end{vmatrix}.
\]

We thus obtain the simple equation

\[
\lambda_1 \frac{\partial G}{\partial \lambda_1} + \lambda_3 \frac{\partial G}{\partial \lambda_3} + G = 0
\]

for the determination of \( G \) as a function of the variables \( \lambda_1 \) and \( \lambda_3 \). Now it is readily seen (cf. sec. 4) that the general solution of (17) is given by \( G = A(\lambda_3/\lambda_1)/\lambda_1 \), where \( A \) is an arbitrary differentiable function of the ratio \( \lambda_3/\lambda_1 \). But from (16) we have \( \lambda_3/\lambda_1 = \lambda_3*/\lambda_1* \). Hence, from the first equation (16), we get \( \lambda_1* = A(\lambda_3*/\lambda_1*) \) as the yield condition. Putting

\[
B \left( \frac{\lambda_3}{\lambda_1} \right) = \left( 1 - \frac{\lambda_2}{\lambda_1} \right) A \left( \frac{\lambda_3}{\lambda_1} \right),
\]

we obtain a modified form of the yield condition and associated stress-strain relations as follows:

yield condition: \( \lambda_1* - \lambda_3* = B \left( \frac{\lambda_3*}{\lambda_1*} \right) \),

stress-strain relations: \( s_{\alpha\beta} = \left( B(\lambda_3/\lambda_1) \right) \epsilon_{\alpha\beta}. \)

Hence we can choose \( B(\lambda_3*/\lambda_1*) \) to be an arbitrary differentiable function of the ratio \( \lambda_3*/\lambda_1* \) in the yield condition (18), and after the choice of the function \( B \) has been made, the associated stress-strain relations will be given by (19). If we take, in particular, \( B = 2k \), where \( k \) is a material constant, we thus obtain

\[
\lambda_1* - \lambda_3* = 2k;
\]

\[
s_{\alpha\beta} = \frac{2k \epsilon_{\alpha\beta}}{\lambda_1 - \lambda_3}.
\]

The first of these equations is the Tresca yield condition, and the second set constitutes the corresponding stress-strain relations.

1 If the assumption of incompressibility were not made, we would define \( \epsilon \) as the deviation of the rate of strain tensor. This would necessitate no modification of the discussion or results of this note.

2 From eq. (2) we have \( \lambda_i^4 = \xi \lambda_i^2 + \tau \lambda_i/3 \) for \( i = 1, 2, 3 \). Taking \( i = 1, 2, 3 \) successively and adding corresponding members of these equations, we immediately obtain the first relation (4). The second and third relations (4) can be deduced in a similar manner.


4 It is assumed that all invariants under consideration are continuous and have continuous first partial derivatives with respect to their arguments. This permits the construction of functional determinants and the use of the implicit function theorem.