One of the main problems in the theory of distributions is the study of the convolution equation

\[ W \ast S = T, \quad (1) \]

where \( W \) is a given distribution of compact carrier. In a previous note\(^1\) we showed that if \( W \) is a punctual distribution (i.e., if eq. (1) is a partial differential-difference equation with constant coefficients), then for every distribution \( T \) we can find a distribution \( S \) such that equation (1) is satisfied. In this paper we shall characterize completely all distributions \( W \) of compact carrier such that equation (1) has a solution in \( \mathcal{D}' \) for every distribution \( T \);\(^2\) if \( W \) has this property, then we say that \( W \) is completely inversive.

We shall show that \( W \) is completely inversive if and only if the Fourier transform \( \mathcal{F}(W) \) is slowly decreasing.

Definition: A function \( J \in E' \) is said to be slowly decreasing if there exist positive numbers \( a, b, c \) such that for each point \( x \in \mathbb{R} \) we can find a point \( y \in \mathbb{R} \) with

\[ |x - y| \leq a \log (1 + |x|), \quad (2) \]
\[ |J(y)| \geq b/1 + |y|^c. \quad (3) \]

We shall also show that \( W \) is completely inversive if equation (1) has a solution \( S \in \mathcal{D}' \) for \( T = \delta \) (the Dirac measure); such an \( S \) is an elementary solution for \( W \). This settles a question posed by L. Schwartz.\(^4\)

Our main result is

**Theorem 1.** For \( W \in \mathcal{E}' \) the following properties are equivalent:

(a) \( \mathcal{F}(W) \) is slowly decreasing.
(b) \( W \ast f \mapsto f \) is a continuous linear map of \( W \ast \mathcal{D} \) into \( \mathcal{D} \).
(c) \( W \ast \mathcal{D}' = \mathcal{D}' \).
(d) There exists an elementary solution for \( W \).
(e) \( W \ast U \mapsto U \) is a continuous linear map of \( W \ast \mathcal{E}' \) into \( \mathcal{E}' \).

We give an outline of the proof; the details will appear elsewhere.

Outline of proof: We prove Theorem 1 by the usual chain of implications: (a) implies (b), (b) implies (c), etc. Of all the statements we have to prove, the implication "(a) implies (b)" is the most difficult. The proof of this uses our previous explicit description\(^1\) of the topology of the Fourier transform of \( \mathcal{D} \), together with a form of the minimum modulus theorem for entire functions of exponential type.\(^6\)

(b) implies (c): This is a simple consequence of the Hahn-Banach theorem.
(c) implies (d): This is a triviality.
(d) implies (e): This can be proved without much difficulty from topological arguments.\(^7\)

(e) implies (a): Suppose that \( W \) is not slowly decreasing, and call \( J = \mathcal{F}(W) \). It is a simple consequence of the theory of quasi-analytic functions that there exists
a function \( G \in \mathcal{D} \) which satisfies
\[
G(x) = O(\exp (|x|^{-1})).
\] (4)

By taking suitable multiples of translates of \( G \), we can produce a set \( B \) of functions in \( \mathcal{E}' \) such that \( JB \) is bounded in \( \mathcal{E}' \) but \( B \) is not bounded in \( E' \). The construction of \( B \) depends on the explicit description of the bounded sets of \( E' \). The existence of \( B \) shows that \( W \ast U \to U \) cannot be a continuous map of \( W \ast \mathcal{E}' \) into \( \mathcal{E}' \), which is the desired result.

The implication "(c) implies (b)" is

**Corollary.** If \( W \) has an elementary solution, then \( W \) is completely invertible.

**Theorem 2.** The statements of Theorem 1 are also equivalent to the following:

\( (b') \) \( W \ast f \to f \) is a continuous linear map of \( W \ast \mathcal{D}_P \) into \( \mathcal{D}_P \).

\( (c') \) \( W \ast \mathcal{D}_P' = \mathcal{D}_P' \).

\( (b^*') \) \( W \ast f \to f \) is a continuous linear map of \( W \ast \mathcal{D}_X \) into \( \mathcal{D}_X \).

\( (c^*') \) \( W \ast \mathcal{D}_X' = \mathcal{D}_X' \).

Theorem 2 is proved in essentially the same manner as the corresponding statements in Theorem 1.

**Theorem 3.** The statements of Theorem 1 are also equivalent to

\( (f) \) \( W \ast \mathcal{E} = \mathcal{E} \).

It is clear that \( (e) \) implies \( (f) \). The converse can be proved without difficulty from the closed graph theorem.\(^{11}\)

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2 See L. Schwartz, *Théorie des distributions*, Vols. 1 and 2 (Paris, 1950-1951). We shall use the following notation: \( \mathcal{D} \) is the space of indefinitely differentiable functions of compact carrier on Euclidean space \( R \) of dimension \( n \). \( \mathcal{D}' \), the dual of \( \mathcal{D} \), is the space of distributions on \( R \). \( \mathcal{E} \) is the space of indefinitely differentiable functions on \( R; \mathcal{E}' \), the dual of \( \mathcal{E} \), is the space of distributions of compact carrier on \( R \).

3 \( \mathcal{E} \), is the Fourier transform of \( \mathcal{E} \); \( \mathcal{D} \) is the Fourier transform of \( \mathcal{E} \).

4 It is incorrectly stated by L. Schwartz (*op. cit.*) that this result for \( n = 1 \) can be proved by the methods of the theory of mean-periodic functions. Professor Schwartz has kindly informed me that his proof only shows that \( W \ast \mathcal{D}_P' = \mathcal{D}_P \) if \( W \) has an elementary solution (see Theorem 2 below). The result of Schwartz was extended to arbitrary \( n \) by Malgrange in his thesis at Paris (to appear in *Ann. Inst. Fourier*).

5 \( W \ast \mathcal{D} \) is the space of all \( W \ast f \) for \( f \in \mathcal{D} \) with the topology induced by \( \mathcal{D} \). A similar notation is employed for other spaces.


7 The proof is similar to the proof of Theorem 4 of my paper "Solutions of Some Problems of Division. II," *ibid.*, pp. 286-292. A different proof is given by Malgrange in his thesis.


11 The details are carried out in Malgrange's thesis.