UNIQUENESS OF THE FINE-STRUCTURE CONSTANT

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1. We sketch here a direct and rigorous approach to the dynamics of fields of elementary particles. We hope at a later date to deal with specific interactions. In the present note it is deduced that under realistically general circumstances the coupling constant in a fully covariant interaction theory must be uniquely determined.\(^1\)

The main novelty in this work is in its formulation of dynamics exclusively in terms of group representations, without the use, in principle, of Lagrangians or partial differential equations. This clarifies the covariance of the theory and leads in particular to a simple, unique method of going from a theoretically given particle to a corresponding field.

2. As is well known, there is associated with a given Lorentz-invariant partial differential equation a representation of the Lorentz group.\(^2\) As indicated in part below, it appears that actually all the operationally significant physics of the particle described by such an equation may be extracted from a corresponding group representation. Hence “particle” may be defined provisionally as a group representation. The problem of formulating the structure of a field of such particles is then one of “quantizing” a group representation, rather than a partial differential equation.

The procedure for this, given below, requires only that the representation be unitary and continuous. Consequently, it is trivial to replace the Lorentz group by an arbitrary one in quantizing. In particular, the space-time co-ordinates of a relativistic particle may be included along with the various momenta by taking as the basic group that spanned by the co-ordinates x, y, z, and \(t\), together with the generators of the Lorentz group and the unit operator. The fact that the commutator of any two such operators is a multiple of a third implies that there exists actually a group in the large of which the group described is the infinitesimal form.\(^3\)

3. There are two mathematical features that seem essential for any quantum statistics: creation and annihilation operators, and transference of one-particle displacements to field displacements. Specifically, we may define a quantum statistics \((\mathcal{C}, \Gamma)\) over a (complex) Hilbert space \(\mathcal{H}\) (the “one-particle” space) as consisting of the suitable assignments (1) to each unit vector \(x\) in \(\mathcal{H}\) an operator \(C(x)\) (the operation of “creating a particle with wave function \(x^*\)”) on a Hilbert space \(\mathcal{K}\) (the “field” state space) and (2) to each unitary operator \(\Gamma(U)\) on \(\mathcal{K}\) (one-particle displacement) a unitary operator \(\Gamma(U)\) on \(\mathcal{K}\) (field displacement).

It is physically appropriate to require that \(\Gamma\) give a continuous representation on \(\mathcal{K}\) of the group of all unitary operators on \(\mathcal{K}\). The relation to the occupation-

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number formalism may be seen through the use of the infinitesimal representation 
\(d\Gamma\) associated with \(\Gamma\), which is defined as taking any self-adjoint operator \(A\) on 
\(\mathcal{K}\) into the self-adjoint generator of the one-parameter group \([\Gamma(e^{itA}); -\infty < \ t < \infty]\). The “number of particles with wave functions in the manifold \(\mathcal{M}\)” is 
then \(d\Gamma(P)\), where \(P\) is the projection with range \(\mathcal{M}\). It can be shown that for 
any representation \(\Gamma\) the proper values of such operators are integral and that they 
are non-negative in accordance with the occupation-number interpretation if and 
only if \(\Gamma\) is the direct sum of canonical representations of the unitary group on the 
spaces of covariant tensors over \(\mathcal{K}\), of maximal symmetry types. The 
properties characterizing the creation operator \(C(x)\) are, notably, that it 
takes a state of the field \(y\) in which there are no \(x\)-particles into a state in which 
there is one \(x\)-particle, and that it is related to \(\Gamma\) through the equation. 
\[
\Gamma(U) \ C(x) \ \Gamma(U)^{-1} = C(Ux).
\]
The statistics may be characterized as “elementary” in case no nontrivial invariant 
exclusion principles may be introduced, which may be formulated mathematically 
as the requirement that \(\mathcal{K}\) be irreducible under the joint action of the \(C(x)\) and the 
\(\Gamma(U)\). Under certain further assumptions it may be shown that there exist only 
two elementary quantum statistics, having as state spaces the spaces \(\mathcal{K}^+(\mathcal{K})\) [or 
\(\mathcal{K}^-(\mathcal{K})\)] of all symmetric [or skew-symmetric] covariant tensors over \(\mathcal{K}\), as represen-
tation \(\Gamma^+\) [or \(\Gamma^-\)] the canonical ones, and as creation operators \(C(x)\) operations 
whose action on tensors of a fixed rank is proportional to the symmetrization [or 
skew-symmetrization] of tensor multiplication by \(x\). These correspond to boson 
and fermion fields, respectively.

The “free field” of particles described by the unitary representation \(U\) on \(\mathcal{K}\) of 
the group \(G\) and having, for example, Bose-Einstein statistics may now be defined 
as having the “kinematics” (or, equivalently in the free-field case, “dynamics”) 
given by the representation \(a \rightarrow x^+ (U(a))\) on the state space \(\mathcal{K}^+(\mathcal{K})\), where \(a\) is 
arbitrary in \(G\). The infinitesimal generators of \(G\) are carried by the associated in-
finitesimal representation into the (integrated) field momenta, energy, etc. The 
“field” itself in the usual sense, after averaging with respect to the wave function \(x\), 
is represented by suitable linear combinations of the \(C(x)\) and their adjoints (cf. 
Cook, op. cit., and the reference there to Bohr and Rosenfeld).

4. Suppose, now, that we have given two interacting particles described by the 
unitary representations \(U_1\) and \(U_2\) of the basic group \(G\), on Hilbert spaces \(\mathcal{K}_1\) and 
\(\mathcal{K}_2\). The kinematics of the coupled field may be roughly described as the dynamics 
of the uncoupled joint field, which is, however, a physical fiction. Alternatively, 
it may be described as the function giving the change in the state space of the 
coupled fields induced by a given change of frame in the underlying one-particle 
geometry. To be explicit, we take for definiteness the case when the particles obey 
Bose-Einstein and Fermi-Dirac statistics, respectively; the “kinematics” is then 
defined as the representation \(\Gamma_0\) of \(G\), where \(\Gamma_0(a) = x^+ (U_1(a)) \times \Gamma^-(U_2(a))\) and 
“\(\times\)” denotes the direct product. The infinitesimal form of \(\Gamma_0\) carries the infinitesi-
mal generators of \(G\) into the momenta, energy, etc., of the so-called “noninteract-
ing” combined field.

The dynamics of the coupled field is given by the function \(\Gamma_T\) on \(G\) which de-
scribes how a transformation in \(G\) affects the actual physical field (“\(T\)” for “total,”
i.e., including interaction). By virtue of what may be called “phenomenological” covariance, applicable to any group-invariant theory, field-theoretic or otherwise, \( \Gamma_\tau \) is a representation of \( G \). There is, in addition, the further type of covariance that is specifically field-theoretic and has the effect of relating the dynamics to the kinematics or, in other terms, connects the actual interacting motion with the hypothetical “noninteracting” motion. This “dynamical” covariance requires roughly that the interaction components of the dynamical variables of the coupled fields should be expressions in the fields that are independent of the frame of reference in the underlying one-particle geometry. Neither type of covariance implies the other type.

The simplest connection between \( \Gamma_\tau \) and \( \Gamma_0 \) arises from the fact that the states of the interacting fields are described in terms of the free-field states. This may be formulated by taking the state space of the coupled fields, i.e., the Hilbert space on which the \( \Gamma_\tau (a) \) act, to be \( \mathcal{H}_+(\mathcal{H}_1) \times \mathcal{H}_-(\mathcal{H}_2) \). In itself, however, this is not a significant restriction on \( \Gamma_\tau \), for (i) any representation unitarily equivalent to \( \Gamma_\tau \) gives the same dynamics; (ii) in realistic cases the representation space of \( \Gamma_\tau \) will be denumerably dimensional, as will the spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) and consequently also the space \( \mathcal{H}_+(\mathcal{H}_1) \times \mathcal{H}_-(\mathcal{H}_2) \); (iii) any two Hilbert spaces of the same dimension are related by a unitary transformation. The substance of dynamical covariance is rather that the \( \Gamma_0 (a) \) and the \( \Gamma_\tau (a) \) act on the same space in a suitably related manner. It is useful to formulate this relation in a precise and explicit way.

To arrive at such a formulation, we recall that in a standard version of field theory the interaction momentum-energy vector is expressed as an infinite polynomial in the annihilation and creation operators of the two particles. This fact in itself has little meaning, for any operator is in a formal sense such a polynomial, as the annihilation and creation operators constitute an irreducible set. However, the transformation properties of the expressions are significant, specifically the fact that they are covariant under changes of the Lorentz frame. This means, for example, that if a polynomial such as \( \Sigma_{i\alpha} r_{i\beta} C^{(1)}(e_i) C^{(j)}(f_j) C^{(k)}(f_k) \) is a term in the expression for the interaction energy \( I(P) \), where \( P \) is the self-adjoint generator of translation in time, the \( [e_i] \) and \( [f_j] \) form orthonormal bases for \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), respectively, and the superscripts on the creation operators designate the particle, then the expression \( \Sigma_{i\alpha} r_{i\beta} C^{(1)}(U_i(a) e_i) C^{(j)}(U_j(a) f_j) C^{(k)}(U_k(a) f_k) \) is a term in the interaction component of the dynamical variable \( I(aP \alpha^{-1}) \) of the field corresponding to the generator \( aP \alpha^{-1} \) of the Lorentz group, for an arbitrary Lorentz transformation \( a \). Now because of the transformation property of \( C(x) \) noted above, the transformed term may also be expressed as \( \Gamma_0 (a) I(P) \Gamma_0 (a)^{-1} \). Thus the Lorentz invariance of the expression for the interaction momentum-energy vector is equivalent to the equations \( \Gamma_0 (a) I(P) \Gamma_0 (a)^{-1} = I(aP \alpha^{-1}) \), where \( I(P) \) denotes the interaction component of the dynamical variable of the coupled field corresponding to the generator \( P \) of a translation in space-time, and \( a \) is an arbitrary Lorentz transformation.

In general, if \( X \) is any infinitesimal generator of the group \( G \) describing the one-particle geometry, the interaction component \( I(X) \) of the corresponding operator for the coupled fields may be unambiguously defined as \( d\Gamma_0 (X) - d\Gamma_\tau (X) \), where \( d\Gamma_0 \) and \( d\Gamma_\tau \) are the infinitesimal representations associated with \( \Gamma_0 \) and \( \Gamma_\tau \), respectively, and the negative of the customary interaction term is used, for reasons
that appear shortly. The discussion in the preceding paragraph leads to the following:

**Definition:** The total motion $\Gamma_\tau$ is *dynamically covariant* in case, for any transformation $a$ of the basic one-particle group $G$ and any infinitesimal generator $X$ of $G$, the interaction component $I(X)$ of the dynamical variable of the coupled fields corresponding to $X$ transforms as follows:

$$I(aXa^{-1}) = \Gamma_\tau(a)I(X)\Gamma_\tau(a)^{-1}. $$

It can be seen purely mathematically that dynamical covariance implies (and in the case of a connected group is implied by) the equations $I([Y, X]) = i[I(X), I(Y)]$, for any generators $X$ and $Y$ of $G$, where the bracket denotes the commutator. This means that $I$ must give an infinitesimal representation of $G$.

5. Invariance under gauge transformations and charge conjugation are, broadly speaking, special cases of the types of covariance already treated. Suitable gauge invariance for coupled fields may be defined as the specialization of the dynamical covariance conditions to the case when the group element $a$ inducing the inner automorphism is in the subgroup whose infinitesimal generators are the space-time co-ordinates $x, y, z$, and $t$. The covariant elimination of the longitudinal waves in the quantization of the electromagnetic field is automatic from the restriction of the state space involved in obtaining a unitary representation.

Invariance under charge conjugation is the special case of dynamical covariance in which the transforming element $a$ is time reversal. To accommodate this element, our earlier definitions are extended by allowing the representation operators to be either unitary or the product of a unitary operator with a conjugation (i.e., a real-linear transformation $J$ such that $J = J^{-1}$ and $J(ix) = -iJ(x)$ for arbitrary vectors $x$). In dealing with these more general representations, it is often convenient to formulate the covariance condition in global terms, as the equation

$$\Gamma_\tau(aba^{-1}) = \Gamma_\tau(a)\Gamma_\tau(b)\Gamma_\tau(a)^{-1}. $$

6. In the standard theory of the coupled electromagnetic and electron-positron fields, the energy-momentum tensor depends in an explicitly linear fashion on the coupling constant $g$ but also depends in an implicit fashion on it, through the dependence of the fields themselves on $g$. However, the commutation and anticommutation relations have no explicit dependence on the coupling constant, so that the implicitly $g$-dependent field can be obtained from a fixed $g$-independent field through transformation by a $g$-dependent unitary operator. Denoting the operators of the fixed field by primes, the relations $I'([X, Y]) = [I'(X), I'(Y)]$ hold for all admissible values of $g$, as such relations are preserved under unitary transformation. The total dependence of $I'(X)$ on $g$ is linear at least when $X$ is a linear or angular momentum, or the energy, generator, by virtue of the usual determination of the basic dynamical variables from the energy-momentum tensor. Substituting an arbitrary linear momentum for $X$ and angular momentum for $Y$, it follows that if there is more than one nonzero admissible value for $g$, then all the interaction linear momenta must vanish. From this it follows from the definition of dynamical covariance that the interaction energy must vanish, which absurdity establishes the uniqueness of $g$. The same reasoning applies to any fully (i.e., both
phenomenologically and dynamically) covariant theory of interaction between two particles, in which the explicit dependence of the basic field dynamical variables on the coupling constant is linear.

7. Besides providing a possible basis for treating other groups $G$ than that of the standard relativistic theory, the present approach may aid in determining theoretically which pairs of elementary particles can admit nontrivial interactions. The existence of a nontrivial representation $\Gamma_T$ satisfying the condition for dynamical covariance imposes a significant restriction on the representations $U_1$ and $U_2$ defining the two elementary particles in question. The same is true of interactions of any finite number of types of elementary particles. The technical problems that arise in the development of this application appear to require a comprehensive mathematical theory of tensor invariants of unitary representations of Lie groups.

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1 The present work is for the most part a summary of a portion of a course given at the University of Chicago during summer, 1955, notes on which are in process of duplication.

2 See, in particular, V. Bargmann and E. P. Wigner, these Proceedings, 34, 211-223, 1948, where further references are given.

3 Difficulties in the definition of satisfactory space-time co-ordinates for elementary relativistic particles are treated at length in the notes mentioned above (n. 1). The "elementarity" of the particle is reflected by the "factorial" character of the corresponding representation, i.e., by the feature that there is no nontrivial subspace invariant under both the representation operators and all the operators commuting with them. However, this definitive aspect of elementarity plays no part in the present note.

4 For the infinite-dimensional case that is realistically relevant, this extension of an essentially well-known finite-dimensional result (see H. Weyl, The Classical Groups [Princeton, N. J.: Princeton University Press, 1939]) is proved by the author in a paper submitted to a mathematical journal. A major part of the determination of the admissible quantum statistics is in a second such paper.

5 These fields have been treated in terms of tensors by J. M. Cook (Trans. Am. Math. Soc., 74, 222-245, 1953). Throughout this note the superscripts "+" and "−" refer to the type of statistics, while the adjoint of an operator is indicated by an asterisk.