ERRATA: ON THE MODULI OF RIEMANN SURFACES

In the article of the above title appearing in these PROCEEDINGS, 41, 236–238, 1955, the following corrections should be made:

On page 236 in the last sentence $2p - 2$ should be replaced by $p$.

The second sentence of the last paragraph on page 237 should read "... not only $S$ but also $D$ ..."

In the same paragraph (p. 238) $C_1$ should read $\tilde{C}_1$.

The last sentence of this same paragraph (p. 238) should read "$T$ is the reflection in $L$ followed by the exchange of the two copies of $D$ (the reflection in the real axis, rotation by $\pi$, translation back); and ..."

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ON THE MODULI OF RIEMANN SURFACES

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In notes¹ in these Proceedings I have shown that, for an algebraic Riemann surface $S$ which is not hyperelliptic, those periods $\pi_{ij}$ of the normal integrals of the first kind, $\int d\xi^n$, whose indices, $i, j$, are those for which the products $d\xi_i d\xi_j$ form a basis of the quadratic differentials $d\xi^2$ on $S$ constitute a set of numerical moduli and also that an analogous theorem holds for plane domains whose doubles are not hyperelliptic.

Now, on a hyperelliptic surface $S$ of genus $p$ the $d\xi^2$ are not all linear combinations of products of simple differentials. In fact, as I will show in a moment, only $2p - 1 d\xi^2$ are of this form, while the remaining $p - 2$ are not. Thus, as pointed out in the first note, a second $S'$ might very well have its periods $\pi'_{ij}$ equal to $\pi_{ij}$ for those indices for which $d\xi_i d\xi_j$ generate the $2p - 1 d\xi^2$ of this form, without being conformally equivalent to $S$. If, however, $S'$ is assumed to be hyperelliptic, too, then I claim that they are indeed conformally equivalent. That is foreshadowed by the fact, already known,² that the hyperelliptic surfaces as a class depend on $2p - 1$ parameters.

THEOREM 1. Let $S$ be a hyperelliptic Riemann surface of genus $p$ with normal integrals of the first kind, $\int d\xi^n$, with periods $\pi_{ij}$ over the retrosections, $\delta_j$, $i, j = 1, \ldots, p$. Let $d\xi_i d\xi_j$, $(i, j) \in (I, J)$, form a basis for those $d\xi^2$ on $S$ which are linear combinations of products of simple differentials. Then, if a second hyperelliptic surface, $S'$, of genus $p$ with periods $\pi'_{ij}$, etc., is such that

$$\pi_{ij} = \pi'_{ij}, \quad (i, j) \in (I, J), \tag{1}$$

then $S'$ can be mapped conformally and 1-1 on $S$.

To establish Theorem 1, I need merely show the slight modifications of the proof of Theorem 1 of the first note required here.

First of all, a hyperelliptic surface is, by definition, one which admits a representation as the two-sheeted surface of $y^2 = f(x)$, where $f(x)$ is a polynomial of degree $2p + 2$, if $x = \infty$ is not a branch-point, and $2p + 1$ if it is. Then the $2p - 2$ simple differentials of the first kind, i.e., everywhere finite, are

$$\frac{dx}{y}, \frac{x dx}{y}, \ldots, \frac{x^{p-1} dx}{y}. \tag{2}$$


² Hölder, op. cit.

³ N. Fine and G. Schweigert, "On the Group of Homeomorphisms of an Arc," Ann. Math. (to appear). This theorem should be a prototype for many others. Analogous theorems for other topological spaces will be called "Fine-Schweigert theorems." The author has established the Fine-Schweigert theorem for the disk and will give its proof in another paper in this series.
All possible products of two of these are the $2p - 1$ finite quadratic differentials
\[
\frac{dx^2}{y^2}, \frac{dx^2}{y^2}, \ldots, \frac{x^{2p-1} dx^2}{y^2}.
\]
(3)

But, besides (3), there are still the $p - 2$ $dt^2$ which are not products of (3),
\[
\frac{dx^2}{y^2}, \frac{dx^2}{y^2}, \ldots, \frac{x^{p-2} dx^2}{y^2}.
\]
(4)

Another way to distinguish (3) from (4) is to observe that $S$ and $S'$ necessarily admit the conformal involutions
\[
T: \ y \to -y \quad \text{and} \quad T': \ y' \to -y',
\]
i.e., the exchange of the two sheets. Under $T$, (3) are invariant; (4) are not.

Now, in minimizing the Dirichlet functional $J(\phi)$ among mappings $\phi$ of $S$ onto $S'$, let the $\phi$ be confined to those which are not only 1-1 and deformable into a fixed $\phi_0$ but which also admit $T$, i.e., for which $\phi(TP) = T'\phi(P)$, $P \in S$. To put it another way, $\phi$ must carry symmetric points into symmetric points.

To express this property of admissible mappings analytically, I observe that, if $z$ is a local parameter at $P \in S$, then it can also be used at $TP$ simply by projecting one sheet on the other (at branch-points $P = T(P)$). Similarly, $w$ at $P' = \phi(P)$ can be used at $T'P'$. Thus, if one rewrites $P' = \phi(P)$ as
\[
w = \phi(z, \bar{z}),
\]
then (5) holds both for $P$ and $P'$ and $TP$ and $T'P'$. Hence in the decomposition
\[
dw \, d\bar{w} = b \, dz \, d\bar{z} + 2 \, \text{Re} \, (a \, dz^2)
\]
(6)
one sees that a $dz^2$ has the same value at $TP$ as at $P$, i.e., must be a linear combination of (3) alone and hence of $dz_1', dz_2'$, but not of (4).

Hypothesis (1), as in the first note, then implies that a $dz^2$ is orthogonal to every $dz_1', dz_2'$ and hence to (3) and, therefore, is identically zero, so that the conclusion follows.

**Theorem 2.** Let $D$ be a plane domain bounded by nonintersecting Jordan curves $C_i$, $i = 1, \ldots, n$, having harmonic measures $\omega_i$, $i = 1, \ldots, n - 1$, and associated analytic functions $w_i = \omega_i + i\mu_i$. Let the double, $S$, of $D$ be hyperelliptic, and let $(\partial w_i/\partial z) (\partial w_j/\partial z) \, dz^2$, $(i, j) \in (I, J)$, form a complex-independent basis on $S$ for the $d\bar{z}^2$ of the form $dz_1' \, dz_2'$ and thus a real-independent basis on $D$. Let the period of the $i$-th harmonic measure of $D$ over $C_i$ be $P_{ij}$. If a similar domain $D'$ with $\omega'_i$ etc., is such that
\[
P_{ij} = P'_{ij}, \quad (i, j) \in (I, J),
\]
(7)
then $D'$ can be mapped conformally and 1-1 on $D$.

This follows from the second note in the same fashion in which Theorem 1 follows from the first note. In fact, if $S$ is hyperelliptic, then not only $S$ but also $S$ admits a conformal involution $T$, as shown in Lemma 1 of the second note. Hence the mappings of $D$ on $D'$ can again be chosen to admit $T$. $T$ can be visualized by mapping $D$ on the upper half-plane so that $C_n$ goes into the real axis, while $C_i$,
i = 1, \ldots, n - 1, go into circles, \( C_i \), where the center of \( C_1 \) is fixed and the center of \( C_2 \) is constrained to lie on a vertical line \( L \) through the center of \( C_1 \). \( T \) is the reflection in \( L \), and the set of \( D \) admitting it consists of those for which the centers of all \( C_i \) lie on \( L \), thus depending on \( n - 2 \) centers and \( n - 1 \) radii = \( 2n - 3 = 2p - 1 \) parameters.

1 On the Transcendental Moduli of Algebraic Riemann Surfaces" (hereafter referred to as the "first note"), these PROCEEDINGS, 41, 42, 1955; "On Moduli in Conformal Mapping" (hereafter referred to as the "second note") these PROCEEDINGS, 41, 176, 1955. The notation in these notes is carried over intact.


3 That is, those which admit \( T \) and are also of the type having hyperelliptic double—obviously there are others admitting \( T \).

NONANALYTIC FUNCTIONS OF ABSOLUTELY CONVERGENT FOURIER SERIES

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We denote by \( A \) the set of all complex-valued functions \( f \) of the form

\[
f(\theta) = \sum_{-\infty}^{\infty} c_n e^{int} \quad (-\pi < \theta \leq \pi)
\]

with \( \sum |c_n| < \infty \). The subset \( A^+ \) of \( A \) consists of all \( f \) in \( A \) for which \( c_n = 0 \) whenever \( n < 0 \). With every \( f \) in \( A^+ \) there is associated an absolutely convergent power series

\[
F(z) = \sum_{-\infty}^{\infty} c_n z^n \quad (|z| \leq 1)
\]

such that \( F(e^{i\theta}) = f(\theta) \).

A famous theorem of Wiener\(^1\) and Lévy\(^2\) asserts that, if \( f \) belongs to \( A \) and if \( \phi \) is a function which is analytic at every point of the range\(^3\) of \( f \), then the composite function\(^4\) \( \phi \circ f \) also belongs to \( A \). The principal result of this note (Theorem 2) shows that the conclusion of the Wiener-Lévy theorem may fail to hold even if there is only one point in the range of \( f \) at which \( \phi \) is merely continuous and not analytic.

The special case in which \( \phi \) is the square root leads to the impossibility of certain factorizations in the convolution algebra \( L_1 \).

Our method is geometric and requires no computation. It depends on the Riemann mapping theorem and on the following two relations between rectifiability and absolute convergence of power series on the closed unit disk:

**Theorem 1.** Let \( F(z) = \sum_{0}^{\infty} c_n z^n \) be analytic in \( |z| < 1 \) and continuous on \( |z| \leq 1 \).

(i) If \( \sum |c_n| < \infty \), then \( F \) maps every radius of the unit disk into a rectifiable curve.