A THEOREM ON THE STRUCTURE OF JORDAN ALGEBRAS*

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The purpose of this paper is to fill a gap in Albert’s structure theory\(^1\) of abstract Jordan algebras of any characteristic \(\neq 2\), by proving the following:

**Theorem A.** Let \(\mathfrak{A}\) be a finite-dimensional Jordan algebra over a field \(\Phi\). Assume (1) that \(\mathfrak{A}\) has an identity element \(u\) and (2) that every element of \(\mathfrak{A}\) has the form \(\alpha u + z\), where \(\alpha \in \Phi\) and \(z\) is nilpotent. Then \(\mathfrak{A} = \Phi u + \mathfrak{B}\), where \(\mathfrak{B}\) is a nil subalgebra of \(\mathfrak{A}\).

This result was proved by Albert for special Jordan algebras and was used by him as a key result in the structure theory of semisimple commutative power-associative algebras. A question which was left open in Albert’s work was that of the structure of simple Jordan algebras over an algebraically closed field having only one nonzero idempotent, the identity \(u\).\(^2\) Theorem A, as stated, implies easily that such an algebra is necessarily \(\Phi u\). This result completes the classification of simple Jordan algebras over an arbitrary field. It also permits us to fill several gaps in the representation theory of Jordan algebras which has been developed by the present author.\(^3\)

Our proof will be based on two new concepts—inverses and ternary composition—which appear to be of some intrinsic interest. Our results on these enable us to adapt Albert’s proof of the theorem for special Jordan algebras to the abstract case. An essential step in the proof is a certain ternary identity which we formulated and which has been proved independently by M. Hall and by L. R. Harper, Jr. (their proofs will appear in a forthcoming issue of the *Proc. Am. Math. Soc.*).

1. **Inverses**

A (nonassociative) algebra \(\mathfrak{A}\) is called a *Jordan algebra* if its multiplication satisfies the identities

\[
ab = ba, \quad (a^2b)a = a^2(ba).
\]

(1)

It is always assumed that the characteristic of the base field is \(\neq 2\). If \(\mathfrak{A}\) is a subspace of an associative algebra \(\mathfrak{E}\), and \(\mathfrak{A}\) is closed relative to the composition \(\{ab\} \equiv ab + ba\), then \(\mathfrak{A}\) with this composition as multiplication is a Jordan algebra. Jordan algebras of this type are called *special*.

Identities (1) have a number of well-known consequences,\(^4\) which we list as follows:

\[
((ab)d)c + ((bc)d)a + ((ca)d)b = (ab)(dc) + (bc)(da) + (ca)(db).
\]

(2)

We write \(A(x, y, z)\) for the *associator* \((xy)z - x(yz)\). Then equation (2) can be rewritten as

\[
A(ab, d, c) + A(bc, d, a) + A(ca, d, b) = 0.
\]

(2')

We write \(R_a\) for the linear mapping \(x \mapsto xa\) in \(\mathfrak{A}\). Then

\[
R_aR_bR_c + R_cR_aR_b + R_{(ac)b} = R_aR_{bc} + R_bR_{ac} + R_cR_{ab}
\]

(3)

\[
= R_{bc}R_a + R_{ac}R_b + R_{ab}R_c,
\]

\[\text{140}\]
\[
([R_a R_b] R_c) = R_{A(a, b, c)}, \quad ([X Y] \equiv XY - YX).
\]

We define \(a^1 = a, a^2 = a^{*1}a\). Then \(\mathfrak{A}\) is \textit{power associative} in the sense that \(a^a a^1 = a^{*2}a\). Also, we have

\[
\begin{align*}
R_{a^2} &= 2R_a R_{a^2} - (2R_{a^2} - R_a^2)R_{a^2}, \\
R_{a^n} &= \frac{1}{2} \left\{ (R_a + (R_{a^2} - R_a)^{1/n}) + (R_a - (R_{a^2} - R_a)^{1/n}) \right\}.
\end{align*}
\]

Let \(\mathfrak{A}\) be a special Jordan subalgebra of an associative algebra \(\mathfrak{A}\) with an identity element \(1\). If \(1 \in \mathfrak{A}\), then \(u = \frac{1}{2} 1\) acts as the identity for \(\mathfrak{A}\) relative to the composition \{ab\} = ab + ba. If \(a\) and \(b\) \(\in \mathfrak{A}\) and \(ab = 1 = ba\), then

\[
\{ab\} = 4u, \quad \{aa\}b = 4a, \quad \{|aa\}b\{bb\} = 16u.
\]

Conversely, it is easy to see that these Jordan relations imply that \(ab = 1 = ba\).

This leads to the following definition.

\textit{Definition 1:} Let \(\mathfrak{A}\) be a Jordan algebra with an identity \(u\). Then an element \(a \in \mathfrak{A}\) is called \textit{regular} in \(\mathfrak{A}\) if there exists a \(b \in \mathfrak{A}\) such that

\[
ab = 4u, \quad a^2b = 4a, \quad a^2b^2 = 16u.
\]

The element \(b\) is called an \textit{inverse} of \(a\).

If \(a\) is regular with \(b\) as an inverse, then \(ba = 4u\) and \(b^2a = \frac{1}{4}(a^2b)b^2 = \frac{1}{4}(a^2b^2)b^2 = 4b\). Hence \(b\) is regular with \(a\) as an inverse. If we set \(c = \frac{1}{2}a, d = \frac{1}{2}b\), then the defining relations give \(cd = u, c^2d = c, c^2d^2 = u\). Using these and equation (5), we can establish

\[
c^2d^k = \begin{cases} 
  c^{l-k} & \text{if } l > k, \\
  u & \text{if } l = k, \\
  d^{k-l} & \text{if } l < k.
\end{cases}
\]

These show that the subalgebra \(\Phi\{u, a, b\}\) generated by \(u, a, b\) is a homomorphic image of the group algebra of an infinite cyclic group. In particular, \(\Phi\{u, a, b\}\) is an associative subalgebra of \(\mathfrak{A}\).

We take up next the questions of uniqueness of the inverse. For this purpose we introduce

\textit{Definition 2:} If \(\mathfrak{A}\) is a Jordan algebra, then we define the \textit{Jordan triple product}

\[
\{abc\} \equiv \frac{1}{2}(ab)c + \frac{1}{2}(bc)a - \frac{1}{2}(ac)b, 
\]

so that \(\{aba\} = (ab)a - \frac{1}{2}a^2b = bU_a\), where

\[
U_a = R_a^2 - \frac{1}{2}R_a^3.
\]

If \(\mathfrak{A}\) is a special Jordan algebra, then \(\{abc\} = abc + cba\) in terms of the associative multiplication. Hence \(\{aba\} = 2aba\). A fundamental property of regular elements, which we proceed to establish, is that if \(a\) is regular, then the linear operator \(U_a\) is regular (has an inverse).

\textit{Lemma 1.} Let \(\mathfrak{B}\) be an associative Jordan subalgebra of a Jordan algebra \(\mathfrak{A}\), and let \(R(\mathfrak{B})^*\) be the associative algebra of linear transformations generated by the \(R_b\) (acting in \(\mathfrak{A}\)), with \(b \in \mathfrak{B}\). Then \(R(\mathfrak{B})^*\) contains a nil ideal \(\mathfrak{N}\) such that \(R(\mathfrak{B})^*/\mathfrak{N}\) is commutative.
Proof: We are assuming that if \( a, b, c \in \mathfrak{B} \), then \( A(a, b, c) = 0 \). By formula (4), this implies that \( [R_a^c R_a^b R_a] = 0 \) holds in \( R(\mathfrak{B})^* \). Now consider the relation
\[
2[R_a R_b R_a] = -[R_a R_b] + [R_a R_b] R_a + 2[R_{ab} R_a] R_a.
\]
Take the commutators with \( R_b \). Since this process is a derivation, we obtain
\[
2[R_a R_b R_a] R_a + 2[R_a R_b] R_a R_b = -[R_a R_b] + [R_a R_b] R_a + 2[R_{ab} R_a] R_a + 2[R_{ab}] R_a R_b.
\]
Take commutators with \( R_b \) again, and use the fact that commutators are in the center. This gives
\[
4[R_a R_b]^3 R_b = 4[R_{ab} R_b][R_a R_b].
\]
If we take the commutators with \( R_a \), this gives \( 4[R_a R_b]^3 = 0 \) and \( [R_a R_b]^3 = 0 \). Hence the commutators \( [R_a R_b] \) are nilpotent. Since they are in the center, they generate a nil ideal \( \mathfrak{N} \) in \( R(\mathfrak{B})^* \). Since \( R_a R_b = R_b R_a \) (mod \( \mathfrak{N} \)) and the \( R_a \) generate \( R(\mathfrak{B})^* \), \( R(\mathfrak{B})^*/\mathfrak{N} \) is commutative.

Theorem B. Let \( \mathfrak{B} \) be a Jordan algebra with an identity \( u \), and let \( a \) be a regular element of \( \mathfrak{B} \) with \( b \) as an inverse. Then \( U_a = R_a^2 - \frac{1}{2} R_a \) has an inverse in the algebra of linear transformations of \( \mathfrak{B} \). Moreover, if the characteristic is \( \neq 3 \), then \( U_a^{-1} = \frac{1}{4} U_b \).

Proof: Let \( \mathfrak{B} \) be the subalgebra generated by \( u, a, \) and \( b \), and let \( R(\mathfrak{B})^* \) be the associative algebra generated by the linear transformations \( R_a = 1, R_x \) for \( x \in \mathfrak{B} \). Then \( R(\mathfrak{B})^* \) contains a nil ideal \( \mathfrak{N} \) such that \( R(\mathfrak{B})^*/\mathfrak{N} \) is commutative. If we use relations (8) in \( [R_a b] + 2[R_a R_b R_a] = R_b R_a + 2[R_a R_b] R_a = R_a R_b + 2[R_{ab} R_a] \), we obtain \( 4R_a + 2R_a R_b R_a = R_b R_a + 8R_a = R_a R_b + 8R_a \). Hence
\[
[R_a R_b] = 0, \quad 4R_a = 2R_a R_b R_a - R_a R_b.
\]
By symmetry, \( [R_b R_a] = 0 \). Next, use relations (8) in \( R_e \circ 1 + 2R_a R_b R_a = R_b R_a + 2R_a R_{ab} \), to obtain
\[
16 = R_{a} R_{a^2} + 8R_a R_b - 2R_a R_{ab} R_a.
\]
If we multiply the second part of equations (10) by \( 2R_b \) on the right-hand side and substitute in equation (11), we obtain
\[
16 = R_{a^2} R_a + 4(R_a R_b)^2 - 2R_a R_{a^2} - 2R_a R_{ab}. \tag{12}
\]
If we use the fact that \( R(\mathfrak{B})^*/\mathfrak{N} \) is commutative, equation (12) implies
\[
16 \equiv (R_{a^2} - 2R_a)(R_{a^2} - 2R_a) \pmod{\mathfrak{N}}. \tag{13}
\]
This shows that \( R_{a^2} - 2R_a^2 \) has an inverse mod \( \mathfrak{N} \), and, since \( \mathfrak{N} \) is a nil ideal, it implies that \( R_{a^2} - 2R_a^2 \), a. d. hence \( U_a \) has an inverse in \( R(\mathfrak{B})^* \). We now use the relation \( [R_a x R_a] + [R_x R_a] + [R_x R_a] = 0 \) (cf. Jacobson), with \( x = a^2, y = a, z = b \), and relations (8), to obtain \( [R_a R_b] = 0 \). Since \( R_a = 3R_a R_a - 2R_a^3 \) and \( [R_a R_a] = 0 \), this gives \( 3R_a^2 [R_a R_b] + 6R_a^2 [R_a R_b] = 0 \). If the characteristic is \( \neq 3 \), then \( U_a [R_a R_b] = 0 \), and, since \( U_a^{-1} \) exists, \( [R_a R_b] = 0 \). Since \( a^2 \) is regular with \( \frac{1}{4} b^2 \) as an inverse, a similar argument shows that \( [R_a R_{a^2}] = 0 \). Hence the four elements \( R_a, R_b, R_{a^2}, R_{a^2} \) commute. By equation (6), these and \( 1 \) generate \( R(\mathfrak{B})^* \).
Hence $R(\mathfrak{B})^*$ is commutative, so that we may take $\mathfrak{R} = 0$. Then relation (13) gives $16 = (R_{a1} - 2R_{a2})(R_{a2} - 2R_{a3})$, so that $U_a(\frac{1}{4}U_a) = 1$.

We shall say that $a$ is a zero divisor if there exists a $b \neq 0$ such that $bU_a = 0$. Theorem B shows that if $a$ is regular, then $a$ is not a zero divisor. Now let $a$ be regular, and let $b$ and $b'$ be inverses. Then $bU_a = 2a = b'U_a$, so that $(b - b')U_a = 0$. Hence $b = b'$. This proves the uniqueness of the inverse.

If $\mathfrak{A}$ is an associative Jordan algebra (commutative associative algebra) with an identity, then $a$ is regular (a zero divisor) in $\mathfrak{A}$ if and only if $a$ is regular (a zero divisor) in the usual associative sense. Now suppose that $\mathfrak{A}$ is arbitrary and that $a$ is an algebraic element of $\mathfrak{A}$. Then it is well known that $a$ is either regular or a zero divisor in the associative algebra $\Phi[u, a]$ generated by $a$ and $u$. Hence $a$ is either regular or a zero divisor (in the Jordan sense) in $\mathfrak{A}$. In particular, suppose that $\mathfrak{A} = \Phi u + \mathfrak{B}$, where $\mathfrak{B}$ is a nil subalgebra. Then every $a = au + b$, $b \in \mathfrak{B}$, is algebraic, and $a$ has an inverse or is nilpotent in the associative algebra $\Phi[u, a]$, according to whether $\alpha \neq 0$ or $\alpha = 0$. In other words, the elements of $\mathfrak{B}$ are either regular or nilpotent, and $\mathfrak{B}$ is just the set of nonregular elements of $\mathfrak{A}$.

2. JORDAN TRIPLE PRODUCT IDENTITIES

In this section we list some useful properties of the Jordan triple product $\{abc\} = \frac{1}{2}(ab)c + \frac{1}{2}(bc)a - \frac{1}{2}(ac)b$. Evidently, $\{abc\}$ is trilinear, that is, is linear in any one of the arguments if the other two are fixed. Also, we have

$$\{abc\} = \{cba\},$$

$$\{abc\} + \{bac\} = (ab)c.$$  (15)

The last has the consequence

$$\{bac\} - \{acb\} = A(a, b, c).$$  (16)

We now list

$$\{aa^*a\} = \frac{1}{2}a^{*+2},$$  

$k \geq 1$,  (17)

$$\{a[ba^*]\}a = \frac{1}{2}[a^{*+1}ba^{*+1}],$$  

$k \geq 1$,  (18)

$$\{aba\}b = a\{bab\},$$  (19)

$$\{(ca)ba\} - \frac{1}{2}\{a(bc)a\} = \frac{1}{2}\{aba\}c,$$  (20)

$$\{(ab)a)b = \frac{1}{4}a^2b^2 + \frac{1}{2}\{ba^2b\} + \{aba\}b,$$  (21)

$$\frac{1}{2}\{ab^2a\} + \frac{1}{2}\{ba^2b\} + a\{bab\} = (ab)^2.$$  (22)

Proofs: Equation (17): $\{aa^*a\} = \frac{1}{2}a^{*+2} + \frac{1}{2}a^{*+2} - \frac{1}{2}a^{*+2} = \frac{1}{2}a^{*+2}$. Equation
(18): Using equation (6), we obtain

\[(R_a)^2 = \frac{1}{4} \left\{ R_a + (R_a^2 - R_a^2) \right\} + 2^k \frac{1}{4} \left\{ R_a - (R_a^2 - R_a^2) \right\}^2 + \frac{1}{2} (R_a^2 - (R_a - R_a^2))^k \]

\[= \frac{1}{2} R_a 2^k + 2^k - 1 U_k. \]

Since \( \{ a^2 b^2 \} = b U_a^k \), this gives \( \{ a^2 b^2 \} a = 2^k - 1 b U_a^k \). Hence \( \{ a \{ a^2 b^2 \} a \} = 2^k - 1 b U_a^k + 1 \). Equation (19): Subtraction of \( 2A(ab, b, a) + A(a, b, b) = 0 \) from \( 2A(ab, b, a) + A(b, a, a) = 0 \) gives \( 2((ab)a)b - 2(ab)^2 + (ab)^2 a - a^2 b^2 - 2((ab)b)a + 2(ab)^2 - (a^2)b + a^2 b^2 = 2((ab)\ a - (b^2)b) - 2((ab)b) + a^2 b^2 = 0. \) Hence \( \{ aba \} b = \{ bab \} a \). Equation (20): \( 2 \{ \{ ca \} ba \} - \{ a(bca) \} - \{ aba \} c = ((ca)b)a + (ba)(ca) - ((ca)\ a) - (bc)\ a + \frac{1}{2} (bc)\ a^2 - ((ba)c) + \frac{1}{2} (ba^2) c = A(b, ca, a) - A(\ a, c, a) - A(b, a, a) + \frac{1}{2} (a^2, b, c) \)

The following identity will be proved in forthcoming papers by M. Hall and L. R. Harper, Jr.: \( \{ aba \}^2 = \{ a \{ ba^2 \} a \} \).

If we linearize relation (23) with respect to \( b \), we obtain \( \{ aba \} \{ aca \} = \{ a \{ ba^2 \} a \} \), which shows that the image space \( H \ U_a \) of \( H \) is a subalgebra of \( \{ aba \} \).

Though we shall not require these, we list two more identities which can easily be established for special Jordan algebras:

\[ \{ \{ aba \} b \{ abc \} \} = \{ \{ aba \} b \{ abc \} \} \]

\[ \{ aba \} c \{ aba \} = \{ a \{ [ aca ] b \} a \} \]

We conjecture that every free Jordan algebra is special. This would mean that every Jordan algebra is a homomorphic image of a special one and that every identity valid for all special Jordan algebras is valid for all Jordan algebras. However, at the present time we do not know whether relations (24) and (25) hold for all Jordan algebras.

3. Proof of Theorem A

**Lemma 2.** Let \( H \) be a Jordan algebra with an identity \( u \), and let \( z \) be a nilpotent element of \( H \). Then \( \{ zab \} \) is not regular for any \( a \in H \).
Proof: We may assume that $z^{-1} \neq 0$, $z' = 0$ for $r \geq 2$. Consider $\{zaz\}z'^{-1}\{zaz\} = (z'^{-1}\{zaz\})\{zaz\} - z'^{-1}\{zaz\}^2$. We have $z'^{-1}\{zaz\} = a(R_z - \frac{1}{2}R_z)R_z^{-1} = a\left(-\frac{1}{2}R_z + R_zR_z\right) = 0$. Similarly, since, by relation (23), $\{zaz\}^2$ has the form $\{zaz\}z^{-1}\{zaz\}$.

Consider $\{zaz\} = a(R_z - \frac{1}{2}R_z)R_z^{-1} = a(\frac{1}{2}R_z + R_zR_z) = 0$. Hence $\{zaz\}z^{-1}\{zaz\} = 0$. Since $z'^{-1} \neq 0$, this implies that $U_w$ has no inverse for $w = \{zaz\}$. Hence, by Theorem B, $\{zaz\}$ is not regular.

Suppose, now, that $\mathfrak{A}$ is a Jordan algebra satisfying the hypotheses of Theorem A. Let $\mathfrak{C}$ be a subalgebra of the form $\Phi + \mathfrak{B}$, where $\mathfrak{B}$ is a nil subalgebra. Suppose that $\mathfrak{C} \neq \mathfrak{A}$. It is known that the associative algebra generated by the elements $R_z, b \in \mathfrak{B}$, is nilpotent. This implies that there exists an element $x \in \mathfrak{C}$ such that $xb \in \mathfrak{C}$ for every $b \in \mathfrak{B}$. By subtracting a suitable multiple of $u$ from $x$, we may suppose that $x$ is nilpotent.

Lemma 3. $\{xbz\}, xb^2, x^2b^2, \{xb^2x\} \in \mathfrak{B}$, and $\{xbz\}x, b'\{xbz\} \in \mathfrak{C}$, for every $b, b' \in \mathfrak{B}$.

Proof: $\{xbz\} = (xb)b - \frac{1}{2}xb^2 \in \mathfrak{C}$. By Lemma 2, $\{xbz\}$ is not regular. Hence $\{xbz\} \in \mathfrak{B}$. Since $xb \in \mathfrak{C}$, and $\mathfrak{B}$ is an ideal in $\mathfrak{C}$, $(xb)b \in \mathfrak{B}$. Hence $xb^2 \in \mathfrak{B}$. Set $xb = \beta u + b', b' \in \mathfrak{B}$. Then, by relation (22), $(xb)^2 = \frac{1}{2}\{xb^2x\} + \frac{1}{2}\{xb^2b\} + x\{xbz\} \in \mathfrak{C}$. Since $x\{xbz\} \in \mathfrak{C}$, this gives

$$\{xb^2x\} + \{xb^2b\} \in \mathfrak{C}. \quad (26)$$

Also, $x(b^2x) \in \mathfrak{C} \land (xb)^2 = \{xb^2x\} + \frac{1}{2}x^2b^2$. Hence

$$\{xb^2x\} + \frac{1}{2}x^2b^2 \in \mathfrak{C}. \quad (27)$$

Next, note that $(xb)x = \beta x + b'x = \beta x + c \in \mathfrak{C}$. Also, $xb = \frac{1}{2}\{xbz\} + 2(bz)x$, so that $(xb)b = -2\{xbz\}b + 2((bz)x)b = -2x\{xbz\} + 2(\beta x + c)b = -2x\{xbz\} + 2\beta u + b'' \in \mathfrak{B}$. Hence

$$(xb)b = -2x\{xbz\} + 2\beta u + b'', \quad b'' \in \mathfrak{B}. \quad (28)$$

Since $(xb)b = \{xb^2b\} + \frac{1}{2}x^2b^2$, this shows that

$$\{xb^2b\} + \frac{1}{2}x^2b^2 \in \mathfrak{C}. \quad (29)$$

Relations (26), (27), and (29) imply that $\{xb^2x\}, \{xb^2b\}, x^2b^2 \in \mathfrak{C}$. Since $x$ and $b$ are nilpotent, $\{xb^2x\}$ and $\{xb^2b\}$ are not regular. Hence they are in $\mathfrak{B}$. By equation (28),

$$x\{xbz\} = -\frac{1}{2}(xb)b + \beta u + \frac{1}{2}b'', \quad (30)$$

and, by equation (22),

$$x\{xbz\} = (xb)^2 - \frac{1}{2}\{xb^2x\} - \frac{1}{2}\{xb^2b\} = \beta u + 2\beta b' + (b')^2 - \frac{1}{2}\{xb^2x\} - \frac{1}{2}\{xb^2b\}. \quad (31)$$
Comparison of equations (30) and (31) gives

\[(x^b)b = \{xb^2x\} + \{bx^2b\} + b'', \quad b'' \in \mathcal{B}.
\]

This implies that \((x^b)b \in \mathcal{B}\). Then \(x^2b^2 = 2(x^b)b - 2\{bx^2b\} \in \mathcal{B}\). Since \(\{xb^2b\} \in \mathcal{B}\), \(\{bxb\} x \in \mathcal{C}\). By equation (20),

\[\frac{1}{2} b'(xbx) = \{(b'x)b\} - \frac{1}{2}\{x(b'b)x\}. \quad (32)
\]

We have \(b'x = \gamma u + \delta, \delta \in \mathcal{B}\). Hence

\[\{(b'x)b\} = \gamma\{ubx\} + \{b(x)\} = \frac{1}{2}\gamma bx + \{b(x)\}. \quad (33)
\]

Also, by equation (15), \(\{b(x)\} = \delta(bx) - \{b(x)\} \in \mathcal{B}\). Since \(\{xbx\} \in \mathcal{B}\), linearization shows that \(\{b(x)\} \in \mathcal{B}\). Hence \(\{xbx\} \in \mathcal{B}\), and equation (33) shows that \(\{(b'x)b\} \in \mathcal{C}\). Linearization shows also that \(\{x(b'b)x\} \in \mathcal{B}\). Hence, by equation (32), \(b'(xbx) \in \mathcal{C}\).

**Lemma 4.** If \(\mathcal{A} \neq \mathcal{C}\), then we can find an \(x \in \mathcal{C}\) such that \(x\) is nilpotent and \(xb\) and \(x^2b \in \mathcal{B}\) for every \(b \in \mathcal{B}\).

**Proof:** Choose \(x\) as in Lemma 3. Suppose that \(\{xbx\} \in \mathcal{C}\) for every \(b \in \mathcal{B}\). Since \(x\) is nilpotent, it will follow that \(\{xbx\} \in \mathcal{B}\) for every \(b\). Then \((xb)^2 = \frac{1}{2}(xb^2x) + \frac{1}{2}\{bx^2b\} + \{xb\} b \in \mathcal{B}\). It follows that \(b' = xb \in \mathcal{B}\) for every \(b\) in \(\mathcal{B}\). Hence \((xb)x = xb' \in \mathcal{B}\), and \(bx^2 = 2(xb)x - 2\{xbx\} \in \mathcal{B}\). Thus \(x\) satisfies our conditions if \(\{xbx\} \in \mathcal{C}\) for all \(b\). Suppose, next, that there is a \(b\) in \(\mathcal{B}\) such that \(\{xbx\} \in \mathcal{C}\). Since \(x\) is nilpotent, \(\{xbx\}\) is not regular, and since \(\{xbx\} = bu + z, z\) nilpotent, \(u = 0\) and \(\{xbx\}\) is nilpotent. Also, by Lemma 3, \(\{xbx\} b' \in \mathcal{C}\) for every \(b'\). Hence we may replace \(x\) by \(\{xbx\}\) in our considerations. Since \(\{xbx\} \in \mathcal{C}\) and \(\{xbx\}\) is nilpotent, there exists a \(k\) such that \(y = \{xbx\}^{k+1} \in \mathcal{C}\) but \(y^2 = \{xbx\}^k \in \mathcal{B}\). Moreover, \(y\) has the form \(\{xb^k\}\), where \(b^k \in \mathcal{B}\), as can be seen by induction on \(k\) using identity (23). We have \(yb' \in \mathcal{C}\) for every \(b' \in \mathcal{B}\). We assert that \(yb' \in \mathcal{B}\), and, since \(y^k \in \mathcal{C}\), this will imply that \(y\) satisfies the conditions on \(x\) in the statement of the lemma. Thus let \(\mathcal{B}\) be the subspace of \(\mathcal{B}\) of elements \(b_1\) such that \(yb_1 \in \mathcal{B}\). By Lemma 3, \(\mathcal{B} \supseteq \mathcal{B}^2\). Since \(y^2 \in \mathcal{B}\), \(\mathcal{C} + \mathcal{F}_y\) is a subalgebra of \(\mathcal{B}\). Let \(b, b_1, \mathcal{B}\), and consider \(y(yb_1) = \frac{1}{2}y^2b_1 + \{yby\} \). Since \(y^2 \in \mathcal{B}\), \(y^2b_1 \in \mathcal{B}_2 \subseteq \mathcal{B}_1\). Hence \(\{yby\} \in \mathcal{B}\), but, since it is not regular, \(\{yby\} \in \mathcal{B}\). Hence \(y(yb_1) \in \mathcal{B}\) and \(yb_1 \in \mathcal{B}_1\), by definition of \(\mathcal{B}_1\). This implies that \(\mathcal{B}_1\) is an ideal in \(\mathcal{C} + \mathcal{F}_y\). Since \(y^2 \in \mathcal{B}\), \(y^2 \in \mathcal{C}\); but, being nilpotent, \(y^2 \in \mathcal{B}\). Hence \(y^2 \in \mathcal{B}\). Suppose that \(\mathcal{B} \neq \mathcal{B}_1\), and let \(b \in \mathcal{B}, \mathcal{B}_1\). Then, by replacing \(b\) by a suitable multiple, we may suppose that \(2yb = u + b'\). Hence \(yb' = 2y(yb) - y = y^2b + 2\{yby\} - y\). We wish to show that this is in \(\mathcal{B}\). Since \(y^2b \in \mathcal{B}\), we have to show that \(2\{yby\} - y \in \mathcal{B}\). We square and obtain \(4\{yby\}^2 + y^2 - 4\{yby\} = \{y\}y^2b\{y\}y + y^2 - 4\{yby\}\). Since \(y^2 \in \mathcal{B}\), \(\{byb\} \in \mathcal{B}\), and, by Lemma 3, \(\{yby\} \in \mathcal{B}\). Also, \(y\{yby\} = bR_y^3 - \frac{1}{2}(by^2)y\), and since \(y^2 \in \mathcal{B}\), \(bR_y^3 = \frac{3}{2}bR_yR_y - \frac{1}{2}bR_y^3 \in \mathcal{B}\). Hence \(y\{yby\} \in \mathcal{B}\),
and so \((2\{yby\} - y)^2 \in \mathfrak{B}\). Since \(2\{yby\} - y = yb' - y^2b \in \mathfrak{C}\), this implies that \(2\{yby\} - y \in \mathfrak{B}\). Hence, also, \(yb' \in \mathfrak{B}\), and so \(b' \in \mathfrak{B}_1\). Hence \(2yb \equiv u \pmod{\mathfrak{B}_1}\), and
\[
\left(\frac{1}{2}(u + y + b)\right)^2 = \frac{1}{4}(u + 2(y + b) + 2yb)
\]
\[= \frac{1}{2}(u + y + b) \quad \pmod{\mathfrak{B}_1}.\]

Set \(f = \frac{1}{2}(u + y + b)\), and let \(\mathfrak{D}\) be the subalgebra generated by \(f\). This algebra is associative and has the nil ideal \(\mathfrak{D} \cap \mathfrak{B}_1\). The coset of \(f\) in \(\mathfrak{D}/(\mathfrak{D} \cap \mathfrak{B}_1)\) is idempotent. Hence we may choose an idempotent \(g\) in this coset. Now it is clear that \(u\) and 0 are the only idempotent elements of \(\mathfrak{A}\). Hence \(g = u\) or \(g = 0\).

In the first case, \(u \equiv \frac{1}{2}(u + y + b) \pmod{\mathfrak{B}_1}\), and, in the second, \(0 \equiv \frac{1}{2}(u + y + b) \pmod{\mathfrak{B}_1}\). Either of these implies that \(y \in \mathfrak{C}\), contrary to assumption. We have therefore proved that \(yb \in \mathfrak{B}\) for every \(b \in \mathfrak{B}\), and \(y\) satisfies our conditions.

We may now prove Theorem A. Thus, let \(\mathfrak{C}\) be a maximal subalgebra of \(\mathfrak{A}\) of the form \(\Phi u + \mathfrak{B}\), where \(\mathfrak{B}\) is a nil subalgebra. If \(\mathfrak{C} \neq \mathfrak{A}\), we can find a nilpotent element \(x \in \mathfrak{C}\) such that \(xb\) and \(x^2b \in \mathfrak{B}\) for every \(b \in \mathfrak{B}\). Then \(\mathfrak{C} + \Phi x\) is a subalgebra properly containing \(\mathfrak{C}\), and \(\mathfrak{B}\) is an ideal in \(\mathfrak{B} + \Phi x\) whose difference algebra is a nil algebra. Hence \(\mathfrak{B} + \Phi x\) is a nil algebra, and \(\mathfrak{C} + \Phi x = \Phi u + (\mathfrak{B} + \Phi x)\). This contradicts the maximality of \(\mathfrak{C}\). Hence \(\mathfrak{A} = \mathfrak{C} = \Phi u + \mathfrak{B}\).

As we noted before, Theorem A implies that the only finite-dimensional simple Jordan algebra over an algebraically closed field having only one idempotent \(\neq 0\) is \(\mathfrak{A} = \Phi u\). This and our earlier results imply that the representations of finite-dimensional separable Jordan algebras are all completely reducible. Also, one can determine the irreducible representations using the structure of \(\mathfrak{A}\).

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2. It is shown by Albert, in "A Theory of Power-associative Commutative Algebras," p. 523, that every finite-dimensional simple Jordan algebra has an identity.


4. All of these except equation (6) can be found in Albert's, "A Structure Theory for Jordan Algebras." Equation (6) is due to W. H. Mills, "A Theorem on the Representation of Jordan Algebras," Pacific J. Math., 1, 255–264, 1951. The form \((2')\) of relation (2) is due to R. D. Schafer, to whom I am also indebted for simplifications in the proofs of identities (19)–(22) below.
