

3. The number of independent cycles, c, in a graph with p points and k lines is given by the Euler formula \( c = k - p + 1 \).

4. Actually it is not sufficient to prove, as in Riddell, op. cit., and Riddell and Uhlenbeck, op. cit., that the correction terms in these expansions are small, since the total number of terms grows with p. One must show that the sum of all correction terms is dominated by the leading term for large p and \( k > p \log p \). A rigorous proof is straightforward but is too lengthy to include in this note.

5. The precise range of k is not known. Polya proves that relation (3) is valid if \( |k - 1/2p| < cp \), where c is a positive constant. However, the range is probably larger; in any case, the range certainly covers the majority of graphs.

6. If there is no root of equation (7) which is less than \( z_0 \), then \( T(x) \) will still be convergent, but the radius of convergence would be determined by \( z_0 \) and no general statement about the analytic behavior of \( T(x) \) can be made, since it depends on the behavior of \( S(z) \) near \( z_0 \).

7. For the definition of homeomorphic type see D. König, Theorie der endlichen und unendlichen Graphen (Leipzig, 1936), p. 5.

8. We will call a chain a line sequence in which the inner points are of order 2, while the end-points are of higher order. These end-points, which are of order 3 or higher, we will call the principal points of the homeomorphic type.

9. The symmetry number of a homeomorphic type we will define as the order of the group of permutations of the principal points which leaves the homeomorphic type invariant.

10. Compare the discussion in Section 4 of Paper I.

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**INDICES OF RANK AND OF SINGULARITY ON ABELIAN VARIETIES**

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The present note is an outline of the formalism of the higher indices of singularity introduced by Comessatti on Abelian varieties\(^1\) and their relation to the rank problem on algebraic varieties.

1. Let \( M_n \) be a compact complex manifold; \( \Gamma = \sum \Gamma^{(p, q)} \) be the exterior algebra of closed, complex differential forms on \( M_n \), bigraded according to type; \( H(M_n, C) = \sum H^d(M_n, C) \) be the complex cohomology algebra of \( M_n \); and \( H(M_n, Q) = \sum H^d(M_n, Q) \) be the rational cohomology algebra of \( M_n \) considered as a subalgebra of \( H(M_n, C) \). The de Rham theory establishes an algebra homomorphism \( \rho: \Gamma \to H(M_n, C) \) (onto); the submodule

\[
H^d_s = \rho( \sum_{p + q = d, p, q \leq k} \Gamma^{(p, q)}) \cap H^d(M_n, Q)
\]
will be called the module of d-dimensional cohomology classes of semirank \( k \), and \( b_k^d = \dim H^d\), the corresponding index of singularity of \( M_n \). Let \( \theta_i, \ldots, \theta_k \in H^2, \)

\[ b = b_i^2 \text{ be a \( Q \)-basis for } H^2, \]

and introduce the \( Q \)-homomorphisms \( L_i, \ldots, L_k : H^{d-k} (M_n, Q) \to H^d \) defined by \( L_i \), \( \ldots, L_k(\phi) = \theta_i \cup \ldots \cup \theta_k \cup \phi \). The \( Q \)-submodule \( H^{d-k} \subset H^d \) generated by the images of the homomorphisms \( L_i, \ldots, L_k, 1 \leq i_1, \ldots, i_k \leq b_i^2 \), will be called the module of d-dimensional cohomology classes of rank \( k \), and \( c_k^d = \dim H^{d-k} \) the corresponding index of rank of \( M_n \). If \( (\phi, \psi) \) is a \( Q \)-valued inner product on \( H^d(M_n, Q) \), and \( \Lambda_{i_1, \ldots, i_k} : H^{d-k} (M_n, Q) \to H^d \) is the algebra homomorphism adjoint to \( L_{i_1}, \ldots, L_{i_k} \), then the \( Q \)-submodule \( H^{d-k} \subset H^d \) which is the intersection of the kernels of the homomorphisms \( \Lambda_{i_1, \ldots, i_k} \) will be called the module of totally \( k \)-effective d-dimensional cohomology classes. Clearly \( H^{d-k} = H^{d-k} \oplus H^{d-k} \), so that \( b_i^{d-k} = c_k^d + a_k^d \), where \( a_k^d = \dim H^{d-k} \).

2. These general definitions take particularly simple forms in two cases of note. First, if \( M_n \) is a nonsingular algebraic variety with \( b_i^2 = 1 \), the totally \( k \)-effective cohomology classes of dimension \( d \leq n \) are dual to the ineffective cycles of classes \( \geq k \) in the sense of Lefschetz; thus the indices introduced above then reduce to well-known invariants. Second, if \( M_n \) is a compact complex torus, \( H(M_n, C) \) is an exterior algebra isomorphic to the subalgebra \( \Gamma_0 \) of \( \Gamma \) consisting of differential forms with constant coefficients, the isomorphism being determined by the period matrix \( \Omega \) of the torus, while \( H(M_n, Q) \) is isomorphic to the subalgebra of \( \Gamma_0 \) consisting of differential forms with rational periods. The indices of rank and of singularity can then be expressed directly in terms of the period matrix. In particular, let \( (\sum \sum) = (\Omega/\Omega)^{-1} \), where \( \sum = (\sigma_{ik}) \) is a \( 2n \times n \) complex matrix.

**Theorem 1.** For a complex torus, \( b_i^d \) is the dimension of the \( Q \)-module of skew-symmetric rational forms \( x_{i_1, \ldots, i_d} \) such that

\[
\sum x_{i_1, \ldots, i_d} = 0
\]

for all indices \( i_1, \ldots, i_d \).

If \( M_n \) is an Abelian variety, it is possible to introduce a \( Q \)-valued inner product on \( H^d(M_n, Q) \) obtained from an inner product on \( \Gamma_0 \) defined by a Hermitian metric \( \sum g_{\mu \nu} dx^\mu dx^\nu \) with constant coefficients; let \( \chi_{i_1, \ldots, i_k} \in H^{d-k} (M_n, Q) \) be rational skew-symmetric forms representing the elements \( \theta_{i_1} \cup \ldots \cup \theta_{i_k} \), and define

\[
\xi_{i_1, \ldots, i_k} = \sum g_{\mu_1 \nu_1} \ldots g_{\mu_k \nu_k} \chi_{i_1, \ldots, i_k} \theta_{\nu_1} \ldots \theta_{\nu_k} \cdot \sigma_{\alpha_1 \beta_1} \ldots \sigma_{\alpha_k \beta_k} \sigma_{\mu_1 \nu_1} \sigma_{\mu_2 \nu_2} \ldots \sigma_{\mu_k \nu_k} \theta_{\alpha_1} \theta_{\alpha_2} \ldots \theta_{\alpha_k} \theta_{\beta_1} \theta_{\beta_2} \ldots \theta_{\beta_k}.
\]

**Theorem 2.** For an Abelian variety, \( a_k^d \) is the dimension of the \( Q \)-module of skew-symmetric rational forms \( x_{i_1, \ldots, i_d} \) such that

\[
\sum x_{i_1, \ldots, i_k} = 0
\]

for all indices \( i_1, \ldots, i_k, \ldots, i_d \).

It should be noted that on Abelian varieties, \( b_i^d \) is the d-dimensional index of singularity of the Riemann matrix \( \Omega \) in the sense of Comessatti. 1
Theorem 3. On a nonsingular algebraic variety, $c_n^d$ is the dimension of the $Q$-submodule of $H^d(M_n, Q)$ generated by cohomology classes dual to $(2n - d)$-dimensional rational cycles formed by intersecting $(2n + k - d)$-dimensional rational cycles with $(n - k)$-dimensional algebraic subvarieties of $M_n$.

Thus the index $c_{n-k}^{2n-k}$ in a sense gives the dimension of the set of $d$-dimensional cycles lying on $k$-dimensional algebraic subvarieties of $M_n$. The proof of this result follows from a theorem of Lefschetz, which asserts that a $(2n - 2)$-dimensional cycle is represented by a divisor (effective or not) if and only if its dual cohomology class is represented by a differential form of type $(1, 1)$, and from a theorem of Severi, which asserts that an irreducible algebraic subvariety $V_k \subset M_n$ is the complete intersection of $r - k$ divisors (effective or not) on $M_n$.

3 This corresponds to the Riemann conditions on $\Omega$.
4 F. Severi, Serie, sistemi d'equivalenza e corrispondenza algebriche sulla varietà algebriche (Rome, 1942).

ON DEHN'S LEMMA AND THE ASPHERICITY OF KNOTS

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Everything in this note will be considered from the semilinear point of view; i.e., any 3-manifold will be considered with a fixed triangulation (this is permissible according to Moise’s work), any curve or line will be considered as polygonal, any surface as polyhedral, and so on.

The following theorem was first considered by Dehn, but it was pointed out by Kneser that Dehn’s proof contains a gap.

Dehn’s Lemma. Let $M$ be a 3-manifold, compact or not, with boundary which may be empty, and in $M$ let $D$ be a 2-cell with self-intersections (singularities), having as boundary the simple closed polygonal curve $C$ and such that there exists a closed neighborhood of $C$ in $D$ which is an annulus (i.e., no point of $C$ is singular). Then there exists a 2-cell $D_0$ with boundary $C$, semilinearly imbedded in $M$.

Johansson proved that, if Dehn’s lemma holds for all orientable 3-manifolds, it also holds for all nonorientable ones. We prove that Dehn’s lemma holds for all orientable 3-manifolds, and by a modification of our method we also prove the following theorem.

Sphere Theorem. Let $M$ be an orientable 3-manifold, compact or not, with boundary which may be empty, such that $\pi_2(M) \neq 0$, and which can be topologically imbedded in a 3-manifold $N$, having the following property: The first homology group of any nontrivial (but not necessarily proper) subgroup of $\pi_1(N)$, has an element of infinite order (note in particular that this holds if $\pi_1(N) = 1$). Then there exists a 2-sphere $S$ semilinearly imbedded in $M$, such that $S$ is not homotopic to zero in $M$. 