INTEGRATION OF A DIFFERENTIAL FORM ON AN
ANALYTIC COMPLEX SUBVARIETY

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I. The purpose of this note is to give a precise definition of the operator of integration

\[ t(\varphi) = \int_W \varphi \]  

(1)

for an exterior differential form \( \varphi \) on an analytic complex subvariety \( W \). The problem arises because an analytic complex subvariety in a domain \( D \) of \( \mathbb{C}^n \) (or, more briefly, an analytic set in \( D \)) is not, in general, a manifold. We give (a) an existence theorem for \( t(\varphi) \) and (b) a proof that \( t \) is a closed current in \( D \), that is, \( t(\psi) = 0 \) for the forms \( \psi \) with compact support, which are homologous to zero in \( D \).

II. A set \( W \) is called an analytic set in a domain \( D \) of \( \mathbb{C}^n(z_1, \ldots, z_n) \) if \( M \in D \) has a neighborhood \( (U_M) \cap W \) is defined by the simultaneous equations

\[ f_1(z_1, \ldots, z_n) = 0, \ldots, f_s(z_1, \ldots, z_n) = 0, \]  

(2)

where \( f_k \) is holomorphic in \( (U_M) \); \( M \in W \) is an ordinary point of \( W \) if there exists an analytic one-to-one mapping \( Z = F(z) \) such that \( F [(U_M) \cap W] = F [(U_M)] \cap C^s \), where \( C^s \) is a complex linear subspace. We denote by \( A^q \) a complex \( q \)-vector given by

\[ z_k = z_k' + \sum_i a_k u_i, \quad 1 \leq k \leq n, \quad 1 \leq i \leq q, \]  

(3)

with unitary representation and fundamental form

\[ d\tau_q = \Omega(A^q) = \left( \frac{i}{2} \right)^q (-1)^{q(q-1)/2} du_1 \wedge d\bar{u}_1 \ldots du_q \wedge d\bar{u}_q. \]

If \( s \) is the maximum of the numbers \( q \) such that \( P \) can be an isolated point of \( W \cap A^s \), \( p = n - s \) is the complex dimension of \( W \) in \( P \in W \). We have the decomposition \( W = W^k \), where \( W^k \) is \( k \) complex dimensional in each of its points. If \( W' = W \cap A^q \) is not of dimension zero, the boundary of \( B(P, r) = \left[ \sum_1^q |u_i|^2 < r^2 \right] \) intersects \( W' \). Starting from this property and applying the Kronecker integral to the system (2) after substituting equation (3), where \( z_k' \) and \( a_k' \) are considered as complex parameters, we obtain

**Theorem 1.** If \( W \) is an analytic set in \( D \) and \( \mathcal{M}(z_0) \) is an isolated point of \( A_0^s \cap W \), where \( A_0^s \) is given by \( (a_k)_0 \) and \( z_k' = z_k^0 \) in equation (3), there exists a neighborhood \( (U_M) \) of \( \mathcal{M}, \epsilon > 0 \) and \( \epsilon' > 0 \), such that for \( |a_k^k - (a_k)_0| < \epsilon \), and \( |z_k' - z_k^0| < \epsilon' \), \( U_M \cap W \cap A^s \) is a set of \( l (\leq N(M_0)) \) isolated points, where \( N(M_0) \) is the number of intersections of \( A_0^s \cap W \) in \( M_0 \).

III. **Positive Currents.**—We denote by \( \ast \psi \) the adjoint of a form \( \psi \); \( \ast \Omega(A^s) \) is the fundamental form of the \( A^{n-s} \) orthogonal to \( A^s \).  

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Definition: A current \( t \) is called positive of degree \( q \) (or belongs to \( \Theta_q^+ \)) if \( t \) is homogeneous of degree \( (q, q) \) in \( dz_1, dz_2 \) and if for every complex \( q \)-vector \( A^q \), the distribution \( T[A^q](f) = \int \wedge^* T^n(A^q)f \) is a positive measure. A form \( \varphi \) is called positive of degree \( q \) (or belongs to \( \Phi_q^+ \)) if \( \varphi \) has continuous coefficients and \( \varphi \in \Theta_q^+ \).

Theorem 2. A positive current \( t = (i/2)^{q(q-1)/2} \sum \frac{t(i)}{t(p)}dz_{i_1} \wedge \ldots \wedge dz_{i_q} \) is continuous of order zero; the distributions \( T_{(i_1)}(f) = \int t_{(i_1)}f \wedge \tau^{2n} \) are complex measures: \( T_{(i_1)} = T_{(i_2)} \) and \( T_{(i_2)}(f) \geq 0 \).

Remark A: We put \( |t|_D = \sup |t(\varphi)| \) for \( \varphi \), \( (C^\infty) \), with compact support \( K(\varphi) \subset D \), and \( |\varphi(0_\varphi)| \leq 1 \) for the coefficients of \( \varphi \). Then it is possible to find a system \( \Lambda \) of \( (C^\infty)^2 \) complex \( q \)-vectors \( A^q \), with \( |a_i^k - (a_i^k)_0| < \epsilon \), where \( \epsilon > 0 \) and \( A_0^q[(a_i^k)_0] \) are given, such that we have \( |t|_D \leq k(\Lambda) \sup \{ T[A^q] \} \), where \( k(\Lambda) \)

is a constant depending on \( \Lambda \). The class of the positive currents possesses the following properties:

Theorem 3 (multiplication). If \( t \in \Theta_p^+ \) and \( \varphi \in \Phi_1^+ \), then \( t \wedge \varphi \in \Theta_{p+1}^+ \).

Theorem 4 (division). If \( t \wedge \varphi = 0 \), \( t \in \Theta_p^+ \), and if \( \varphi \in \Phi_1^+ \) satisfies \( \varphi = \psi \neq 0 \) and \( \psi^{p+1} = 0 \) in \( D \), then we have \( t = t_1 \wedge \varphi^2 \), and \( t_1 \in \Theta_{p_1} (t_1 = 0 \text{ if } p < q) \).

Theorem 5. The image of a positive current induced by an analytic and locally one-to-one transformation is a current which is positive.

IV. If \( W \) is the analytic set \( f(z_1, \ldots, z_n) = 0 \) in \( D \), we have the following two expressions for the current \( t(\varphi) \):

\[
(a) \quad t = \frac{2}{\pi} \left[ \int_0^\pi d\theta \log |f| \right]; \quad (b) \quad t = \theta \left[ \frac{1}{2} df \wedge df. \right.
\]

In \( a \), the bracket denotes the positive current relative to the plurisubharmonic function \( \log |f| \). On the other hand, \( b \) is available only for forms \( \varphi \) whose support \( K(\varphi) \) contains only ordinary points of \( W \); in \( b \) the positive current \( \theta \) of degree zero (measure density) is given by an analytic locally one-to-one mapping \( Z = F(z) \) of \( W \) on a complex subspace.

We consider now an analytic set \( W_p \) of complex dimension \( p \) irreducible in \( D \) and denote by \( W_p^\infty \) the connected manifold of the ordinary points of \( W_p \), and we put \( E_1 = W_p^\infty - W_p^\infty \). The definition of the current \( t(\varphi) \) (the integral of \( \varphi \) on \( W_p \)) will be given in three steps: (1) Definition of the positive current \( t_0(\varphi) = \int t(\varphi) \) with support \( K(\varphi) \subset D - E_1 \). (2) Majorization of the norm of \( t_0 \) in a neighborhood of \( M \in W_p^\infty \), for instance in a sphere \( B(M, r) \). (3) Solving of a continuation problem for \( t_0 \) defined in \( D - E_1 \) to obtain \( t \) defined in \( D \).

1. If \( M \in W_p^\infty \), \( t_0 \) is obviously defined by an analytic mapping of \( W_p \) into \( (U_M) \) in a complex subspace.

2. We use the following result, which is a consequence of Theorem 1 and Remark A:

Theorem 6. If \( W \) is an analytic set in \( D \), of complex dimension \( p \) in every \( M \in W \), and \( D_1 \subset D \) is a compact domain in \( D \), there exists a constant \( \lambda(D_1) > 0 \) such that the norm \( |t_0|_M \) of the current \( t_0 \) in the spheres \( B(M, r) \subset D_1 \) satisfies \( |t_0|_M < \lambda(D_1)r^{2p} \).

V. Continuation of a Closed Current.—Given a current \( t \) in a domain \( D - E_1 \) where \( D \subset \mathbb{R}^n(x_1, \ldots, x_n) \) and \( E \) is the subspace \( x_1 = \ldots = x_n = 0 \), a necessary condition that \( t \) be continuability to the forms \( \varphi \) with support \( K(\varphi) \subset D \) is the
convergence of \( \sum t \land \alpha \varphi, K(\alpha_i) \subset D - E \), where \( \alpha_i \) is a partition of unity in \( D - E \). We give now conditions that a closed current \( t \) has a continuation \( \bar{t} \) by a closed current. We denote by \( \alpha_i(x_1, \ldots, x_\ast \) a kernel \( (C^\infty) \), with \( 0 \leq \alpha_i \leq 1 \); \( \alpha_i = 1 \) in a neighborhood \( \omega \), of the origin \( \omega \) \( (x_1 = \ldots = x_\ast = 0) \); \( \alpha_i \) has a compact support \( K_i \); \( \omega \subset K_i \subset \omega' \), and \( \omega' \) tends to \( \omega_0 \) when \( r \to 0 \). We put \( t = \bar{t} - t \), and suppose that \( t \) is homogeneous of degree \( k \).

**Theorem 7.** A necessary and sufficient condition for the existence of \( \bar{t} \) is (a) \( \lim \alpha_i t = 0 \); (b) the existence of the limit \( \tau(\varphi) = (-1)^{(i) t([\alpha_i] \land \varphi)} \), for a sequence of kernels \( \alpha_i \) with properties listed above.

Thus \( \tau \) is a current with support in \( E \) and is homologous to zero; the limit is independent of the sequence \( \alpha_i \); we have \( d\theta = \tau_i, \bar{t} = t + \theta \). If \( \tau = 0 \), we choose \( \theta = 0 \), and we say that \( \bar{t} \) is the simple extension of \( t \). If \( \bar{t} \) exists, \( t \) is continuous of finite order \( q \) in \( K \cap (D - E) \), where \( K \) is a compact domain in \( D \). We put \( |t|_K = \sup |t(\varphi)| \), where \( \varphi \) satisfies \( K(\varphi) \subset K \cap \left( \sum_{i=1}^{n} x_i^2 < r^2 \right) \) and \( |D^{(l)}\varphi_0| \leq 1 \), for each derivative of total order \( \leq q \) of the coefficients of \( \varphi \). We choose \( \alpha_i = \alpha(x_i/r) \) and denote \( (\alpha_i/\partial x_i)(x_i/r) \) by \( \beta_i, (x_i) \), to obtain more precise sufficient conditions:

**Theorem 8.** A sufficient condition for the existence of \( \bar{t} \) is the existence of a finite number \( L \) such that we have \( |t\beta, (x_i)|_K \leq Lr \).

**Corollary.** If \( t \) is continuous of order zero and if \( r^{-1}|t|_K \to 0 \), \( \bar{t} \) is obtained by the simple extension of \( t \) in \( K \subset D \).

VI. Now we consider \( E_1 = W^p - W_\emptyset \) and put \( E_1 = W_1 \cup W_2 \cup \ldots \cup W_s \), where \( W_k \) is the manifold of the ordinary points of \( E_1 \subset (W_1 \cup W_2 \cup \ldots \cup W_{s-1}) \); \( W_k \) is a manifold of complex dimension \( n_k \leq p - k \). In \( M \in W_k \), we consider a neighborhood \( (U_M) \) and an analytic one-to-one mapping \( Z = F(z) \) such that \( F((U_M) \cap W_k) = F((U_M) \cap C^\infty) \) where \( C^\infty \) is a complex subspace. By Theorems 5 and 6 and the corollary to Theorem 8, we obtain the main result of this note:

**Theorem 9.** If \( W^p \) is an analytic set in a domain \( D \) of \( C^\infty \) and if \( W^p \) is of complex dimension \( p \) in each of its points, the positive current of degree \( (p, p) \) defined by

\[ t_0 = \int_{W^p} \varphi \]

(where \( W^p_0 \) is the manifold of the ordinary points of \( W^p \), and \( E_1 = W^p - W^p_0 \) is convergent in \( D - E_1 \). The simple extension \( t \) of \( t_0 \) gives a closed positive current \( t(\varphi) \) on the forms \( \varphi \) with compact support in \( D \).

If \( W \) is not homogeneous \( p \) complex dimensional in \( D \), the current \( t(\varphi) = \int_W \varphi \)

is a finite sum of homogeneous, positive closed currents \( t_k \) in \( D \) whose supports are the components of \( W \) in the decomposition \( W = \cup W_k \), where \( W_k \) is \( k \) complex dimensional in each of its points.

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\(^2\) Cf. G. de Rham, *Variétés différentiables* (Paris: Hermann & Cie, 1955). A current \( t \) is continuous of order \( s \) if \( t(\varphi_\alpha) \to 0 \) for every sequence of forms \( \varphi_\alpha, (C^\infty) \), with compact supports \( K(\varphi_\alpha) \subset K_\delta \), and \( \sup \|D^{(n)}(\varphi_n, t)\| \to 0 \), for every coefficient \( \varphi_n, (t) \) and every derivative \( (\alpha) \) of total order \( \leq s \).