shown with the same substrates by Chance and Williams for mammalian mitochon-
dria.

Also, the value of $2.1 \times 10^{-2}$ moles P/mole $R \times sec$ for $k_{3a\text{ succinate}}$ is in essential
agreement with the value of $2.3 \times 10^{-2}$ moles P/mole $R \times sec$ for the $k_{3a}$ calculated
from kinetic analysis of uptake measurements of control roots utilizing an endo-
genous level of substrate. However, the value of $8.3 \times 10^{-2}$ moles P/mole $R \times sec$ for
either $k_{3a}$ or $k_{3b\text{ -oHB}}$ is much lower than the value for $k_{3a}$ of $1.4 \times 10^{-2}$ moles
P/mole $R \times sec$ which was calculated from uptake measurements of control roots.
A study of the effect on orthophosphate uptake of numerous substrates which re-
duire DPN in the pathways of reducing reactions has resulted in the conclusion
that the $k_{3a}$ of glutamate corresponds most closely to the $k_{3a}$ calculated from up-
take measurements of excised barley roots utilizing an endogenous level of sub-
strate.

CONCLUSIONS

Orthophosphate uptake by barley roots is coupled to sites identical with those for
oxidative phosphorylation during electron transport in mammalian mitochondria.

2. Britton Chance, G. R. Williams, William F. Holmes, and Joseph Higgins, J. Biol. Chem., 217,
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THE NORM FORM OF A RATIONAL DIVISION ALGEBRA*

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Let $D$ be a finite-dimensional associative division algebra over the field $\mathbb{R}$ of all
rational numbers. Then the center $\mathfrak{g}$ of $D$ is an algebraic extension of finite degree
over $\mathbb{R}$, and $D$ is a cyclic algebra of degree $n$ and dimension $n^2$ over $\mathfrak{g}$. We may
then write $D = (\mathfrak{g}, S, \gamma)$, where $\mathfrak{g}$ is a cyclic field of degree $n$ over $\mathfrak{g}$, $\gamma$ is a nonzero
element of \( \mathfrak{f} \), and \( S \) generates the cyclic automorphism group of \( \mathfrak{g} \) over \( \mathfrak{f} \). The general element \( x \) of \( \mathfrak{D} \) now has the form \( x = x_0 + x_1 y + \cdots + x_{n-1} y^{n-1} \), where the \( x_i \) are in \( \mathfrak{g} \), and \( y \) in \( \mathfrak{D} \) has the property \( y^n = \gamma, yz = (\alpha S)y \).

The algebra \( \mathfrak{D} \) now has an isomorphic representation as an algebra of \( n \)-rowed square matrices with elements in \( \mathfrak{g} \). It is the mapping \( x \rightarrow X = X_0 + X_1 Y + \cdots + X_{n-1} Y^{n-1} \), where \( X_i = \text{diag} \{ x_{i0}, x_{i1}, \ldots, x_{iS^n-1} \} \) and

\[
Y = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
\gamma & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}
\]

Then we define the norm form of \( \mathfrak{D} \) to be the determinant

\[
\Delta(x) = |X|
\]

of the representation matrix. It is a homogeneous polynomial of degree \( n \) in the \( n^2 \) co-ordinates in \( \mathfrak{f} \) of the general element \( x \) of \( \mathfrak{D} \), and it permits composition, that is

\[
\Delta(xy) = \Delta(x) \cdot \Delta(y).
\]

The purpose of this note is to provide a proof of the following result.

**Theorem.** Let \( \mathfrak{D} \) be a division algebra of odd prime degree over its center \( \mathfrak{f} \), where \( \mathfrak{f} \) is an algebraic number field. Then the norm form of \( \mathfrak{D} \) is universal, that is, every nonzero element \( \delta \) of \( \mathfrak{f} \) is the norm \( \delta = \Delta(x) \) of an element \( x \) in \( \mathfrak{D} \).

Let us begin by recalling some of the properties of rational division algebras.¹ We let \( \mathfrak{f} \) be the ring of all integers of \( \mathfrak{f} \), so that every prime ideal \( \mathfrak{p} \) of \( \mathfrak{f} \) determines a valuation \( \phi_{\mathfrak{p}} \) of \( \mathfrak{f} \), and a completion \( \mathfrak{f}_{\mathfrak{p}} \) of \( \mathfrak{f} \). The scalar extension \( \mathfrak{D}_{\mathfrak{p}} \) of \( \mathfrak{D} \) by \( \mathfrak{f}_{\mathfrak{p}} \) will then be a total matric algebra for all but a finite number² of valuations \( \phi_{\mathfrak{p}} \). Let us call the primes \( \mathfrak{p} \) for which \( \mathfrak{D}_{\mathfrak{p}} \) is a division algebra the critical primes for \( \mathfrak{D} \). Then it is known³ that a field \( \mathfrak{r} \), of the given odd prime degree \( n \) over \( \mathfrak{f} \), is a splitting field for \( \mathfrak{D} \) if and only if the degree \( n_{\mathfrak{p}} \) of each composite \( (\mathfrak{r}, \mathfrak{f}_{\mathfrak{p}}) \) over \( \mathfrak{f} \) is \( n \) for every critical prime. Thus, if \( \mathfrak{r} = \mathfrak{f}[\xi] \), where \( \xi \) is a root of

\[
f(\omega) = \omega^n + \alpha_1 \omega^{n-1} + \cdots + \alpha_n,
\]

with \( \alpha_i \) in \( \mathfrak{f} \), and \( f(\omega) \) is irreducible in \( \mathfrak{f}_{\mathfrak{p}} \) for every critical prime \( \mathfrak{p} \), it will follow that \( \mathfrak{r} \) splits \( \mathfrak{D} \). Then \( \mathfrak{r} \) will be isomorphic to a subfield \( \mathfrak{f}[x] \) of \( \mathfrak{D} \), \( f(x) = 0 \), \( \alpha_n = (-1)^n \Delta(x) \). It should now be clear that our theorem is equivalent to the property that if \( \alpha_n \) is any nonzero element of \( \mathfrak{f} \), there exist elements \( \alpha_1, \ldots, \alpha_{n-1} \) in \( \mathfrak{f} \) such that \( f(\omega) \) is irreducible in \( \mathfrak{f}_{\mathfrak{p}} \) for any prescribed finite set of prime ideals \( \mathfrak{p} \) of \( \mathfrak{f} \).

However, it will be convenient for us to remain closer to the theory of algebras than this remark might suggest.

There is no loss of generality if we assume that \( \alpha_n \) is an integer of \( \mathfrak{f} \), since \( \alpha_n = -\Delta(x) \) if and only if \( \lambda^\ast \alpha_n = -\Delta(\lambda x) \). We can clearly select \( \lambda \) so that \( \lambda^\ast \alpha_n \) is integral. We shall also select the elements \( \alpha_1, \ldots, \alpha_{n-1} \) to be algebraic integers, so that any root \( \xi \) of \( f(\omega) = 0 \) is always integral. Every element of \( \mathfrak{f}_{\mathfrak{p}} \) has the form \( \alpha = \pi^m \mu \), where \( \pi \) can be taken to be any element of \( \mathfrak{p} \) not in \( \mathfrak{p}^2 \), and \( \mu \) is a unit.

¹ Theorem.
² Properties of rational division algebras.
³ Note.
of the ring \( \mathfrak{B} \) of all integers of \( \mathfrak{F} \). Then \( \lambda \geq 0 \) for every element of \( \mathfrak{F} \). We take \( \mathfrak{B} \) to be a critical prime for \( \mathfrak{D} \) and write

\[
\alpha_n = \pi^k \beta_n, \quad \lambda = nk + t \quad (0 \leq t < n),
\]

where \( \beta_n \) is a unit of \( \mathfrak{F} \). If \( t > 0 \), we select the \( \alpha_t \) so that

\[
\alpha_t \equiv 0 \pmod {\mathfrak{F}^{nk+t}}. \tag{1}
\]

Then \( \alpha_t = \pi^{nk+t} \beta_t \) for integers \( \beta_t \) in \( \mathfrak{F} \), and \( f(\omega) = \omega^n + \pi^{nk} (\beta_1 \pi \omega^{n-1} + \beta_2 \pi^2 \omega^{n-2} + \cdots + \beta_{n-1} \pi^{n-1} \omega + \pi^t \beta_n) \). As we have said, \( f(\omega) \) has leading coefficient 1 and other coefficients in \( \mathfrak{F} \), and so every root of \( f(\omega) \) is integral over \( \mathfrak{F} \). But then \( f(\omega) \) of prime degree is reducible if and only if it has a root \( \rho \) in \( \mathfrak{F} \), and we may write \( \rho = \pi^t \sigma \) for a unit \( \sigma \) of \( \mathfrak{F} \). Clearly \( \pi^{nk+t} \) divides \( \rho^n \), and so \( \nu = k + s \), where we must have \( s > 0 \). Then \( \pi^{t} \) is a factor of \( \pi^n \) as well as of \( \beta_t \pi^p \omega^{-1} \), and so \( \pi^n \) divides \( \pi^t \beta_n \), which is impossible. Thus the congruences (1) make \( f(\omega) \) irreducible in the field \( \mathfrak{F} \) for every critical prime divisor \( \mathfrak{B} \) of \( \alpha_n \) such that the exact power of \( \mathfrak{B} \) dividing \( \alpha_n \) is \( \mathfrak{B}^{nk+t} \), where \( 0 < t < n \).

There remains the case where \( t = 0 \), so that \( \alpha_n = \pi^k \beta_n \), with \( \beta_n \) a unit of \( \mathfrak{F} \). We assume that

\[
\alpha_t \equiv \pi^{nk+1} \gamma_t \pmod {\mathfrak{F}^{nk+n+1}}, \tag{2}
\]

where the elements \( \gamma_t \neq 0 \pmod {\mathfrak{F}} \) will be selected later. Then \( \alpha_t = \pi^{nk+t+1} \beta_t \), where \( \beta_t \) is a unit of \( \mathfrak{F} \), and may be determined modulo \( \pi \). But then \( f(\omega) = \omega^n + \beta_1 \pi^{nk-(n-1)} \omega^{n-1} + \cdots + \beta_{n-1} \pi^{nk-1} \omega + \pi^k \beta_n \) has a root in \( \mathfrak{F} \) if and only if \( \pi^{-nk} f(\omega^k) = g(\omega) = \omega^n + \beta_1 \omega^{n-1} + \cdots + \beta_{n-1} \omega + \beta_n \) has a root in \( F_\mathfrak{B} \). It is known that the algebra \( \mathfrak{D}_\mathfrak{B} = (\mathfrak{B}, T, \pi) \), where \( \mathfrak{B} \) is a cyclic unramified field over \( \mathfrak{F} \) with generating automorphism \( T, \mathfrak{B} = \mathfrak{F}_\mathfrak{B}(h) \) for a root of unity \( h \). The residue-class field \( \mathfrak{F}_\mathfrak{B} = \mathfrak{F} - (\pi) \) is a finite field of \( q \) elements, and \( \mathfrak{F}_\mathfrak{B}(h) \) has degree \( n \) over \( \mathfrak{F}_\mathfrak{B} \); \( h \) is a primitive \((q^n-1)\)st root of unity, \( hT = h^q \). Every unit of \( \mathfrak{F}_\mathfrak{B} \) is known to be a norm in \( \mathfrak{B} \), and we observe that, if

\[
a = \frac{hT}{h} = h^{q-1},
\]

then the norm of \( a \) as an element of \( \mathfrak{B} \) is 1. However, \( a \) is never in \( \mathfrak{F}_\mathfrak{B} \), since otherwise the residue class of \( a \) has period \( q - 1 \), \( h^{(q-1)(q-1)} = 1 \pmod {\pi} \), whereas the period of \( h \) is \( q^2 - 1 > (q - 1)^2 \), since \( q^2 - 1 + \cdots + q + 1 > q - 1 \) when \( q \geq 2 \). But if \( \beta_n = -N(b) \) for \( b \) in \( \mathfrak{B} \), and \( b \) is in \( F_\mathfrak{B} \), then \( \beta_n = -N(ab) \), where \( ab \) is not in \( \mathfrak{F}_\mathfrak{B} \). It follows that \( \beta_n = -N(b) \), where \( b \) generates the cyclic field \( \mathfrak{B} \) of degree \( n \) over \( \mathfrak{F}_\mathfrak{B} \). Then \( b \) is a root of \( g_\mathfrak{B}(\omega) = \omega^n + \sigma_1 \omega^{n-1} + \cdots + \sigma_{n-1} \omega + \beta_n \), and we can take

\[
\beta_t = \sigma_t \pmod {\mathfrak{B}}. \tag{3}
\]

Since \( \mathfrak{B} \) is unramified, the polynomial \([g_\mathfrak{B}(\omega)] = \omega^n + [\sigma_1] \omega^{n-1} + \cdots + [\sigma_{n-1}] \omega + [\beta_n] \), over the residue-class ring \( \mathfrak{D}_\mathfrak{B} \), is irreducible. Then (3) implies that \( g(\omega) \) is irreducible, for a reducible polynomial cannot have an irreducible residue class. Thus the selection (3) makes \( f(\omega) \) irreducible in the cases of critical primes \( \mathfrak{B} \) with \( \alpha_n \) exactly divisible by \( \mathfrak{B}^{nk} \). Since we have a finite number of congruences to be satisfied by the \( \alpha_t \), we can select them so that \( f(\omega) \) is irreducible, as desired.
This completes our proof in the case where $\mathbb{F}$ is an extension of finite degree over $\mathbb{R}$. The result is true, however, even when $\mathbb{F}$ is infinite-dimensional over $\mathbb{R}$. For $\mathcal{D}$ has a basis $u_1, u_2, \ldots, u_m$ over $\mathbb{F}$ and a multiplication table $u_i u_j = \gamma_{ij} u_k$ for algebraic numbers $\gamma_{ij}$, where $m = n^2$. If $\delta \neq 0$ is in $\mathbb{F}$, we let $\mathbb{F}_\delta$ be the field obtained by adjoining the $\gamma_{ij}$ and $\delta$ to $\mathbb{R}$. Then $\mathcal{D} = (\mathcal{D}_0)_{\mathbb{F}}$, where $\mathcal{D}_0 = u_1 u_0 + \cdots + u_m u_0$ is a central division algebra of degree $n$ over $\mathbb{F}_\delta$. The field $\mathbb{F}_\delta$ is a finite algebraic extension of $\mathbb{F}$, and so $\delta = \Delta(x)$, where $x$ is in $D_0$. But $\Delta(x)$ is the norm form of $\mathcal{D}$, and our result is proved.

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1 An exposition of this theory is given as chap. IX of the author's Structure of Algebras.


3 This is a consequence of Theorems 9.31 and 9.23.

4 See Theorem 9.21.


THE RANDOM FUNCTIONS OF COSMIC-RAY CASCADES

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1. Introduction.—We consider the electron-photon cascades of cosmic radiation in the formulation of Bhabha and Heitler and of Carlson and Oppenheimer. We make the simplifying assumptions (used in some of the physical literature) appropriate to the so-called "complete screening" case. Hence the asymptotic results given below cannot be strictly true, since our assumptions hold only over limited ranges of the parameters. Presumably, detailed analytical study of more complicated models would be necessary to determine whether the results have some qualitative validity.

A photon of positive energy $\epsilon$, moving through homogeneous material, has probability $\lambda \ dt + o(dt)$ (henceforth we shall omit the $o$-terms in such expressions) of being transformed in the thickness interval $(t, t + dt)$ into two electrons, positive and negative, which receive respective energies $\epsilon u$ and $\epsilon(1 - u)$, $0 < u < 1$, with probability density $q(u)$. We assume that $q$ has a bounded derivative for $0 \leq u \leq 1$ and is symmetric about $1/2$. An electron loses (by "collision" or "ionization") the deterministic amount of energy $\beta t$ in any interval of length $t$, provided that the energy is not thereby reduced below 0; $\beta$ and $\lambda$ are constants independent of $t$ and of the energy. Also, an electron radiates photons ("Bremsstrahlung") : the probability that an electron of energy $\epsilon$ emits a photon of energy between $\epsilon u$ and $\epsilon(u + du)$ in the interval $dt$ is $k(u) \ du \ dt$, the energy which goes to the photon being subtracted from that of the electron. The usual simplifying assumptions are covered if we take

$$k(u) = \frac{\mu}{u} + k_0(u), \quad \frac{dk_0}{du} \leq c(1 - u)^{-\beta}, \quad 0 \leq u < 1, \quad (1.1)$$