THE RANDOM FUNCTIONS OF COSMIC-RAY CASCADES

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1. Introduction.—We consider the electron-photon cascades of cosmic radiation in the formulation of Bhabha and Heitler and of Carlson and Oppenheimer. We make the simplifying assumptions (used in some of the physical literature) appropriate to the so-called "complete screening" case. Hence the asymptotic results given below cannot be strictly true, since our assumptions hold only over limited ranges of the parameters. Presumably, detailed analytical study of more complicated models would be necessary to determine whether the results have some qualitative validity.

A photon of positive energy $\epsilon$, moving through homogeneous material, has probability $\lambda \, dt + o(dt)$ (henceforth we shall omit the $o$-terms in such expressions) of being transformed in the thickness interval $(t, t + dt)$ into two electrons, positive and negative, which receive respective energies $\epsilon u$ and $\epsilon(1 - u)$, $0 < u < 1$, with probability density $q(u)$. We assume that $q$ has a bounded derivative for $0 \leq u \leq 1$ and is symmetric about $1/2$. An electron loses (by "collision" or "ionization") the deterministic amount of energy $\beta t$ in any interval of length $t$, provided that the energy is not thereby reduced below 0; $\beta$ and $\lambda$ are constants independent of $t$ and of the energy. Also, an electron radiates photons ("Bremsstrahlung"): the probability that an electron of energy $\epsilon$ emits a photon of energy between $\epsilon u$ and $\epsilon(u + du)$ in the interval $dt$ is $k(u) \, du \, dt$, the energy which goes to the photon being subtracted from that of the electron. The usual simplifying assumptions are covered if we take

$$k(u) = \frac{\mu}{u} + k_0(u), \quad \left| \frac{dk_0}{du} \right| \leq c(1 - u)^{-\beta}, \quad 0 \leq u < 1, \quad (1.1)$$

This completes our proof in the case where $\mathfrak{F}$ is an extension of finite degree over $\mathfrak{R}$. The result is true, however, even when $\mathfrak{F}$ is infinite-dimensional over $\mathfrak{R}$. For $\mathfrak{F}$ has a basis $u_1, u_2, \ldots, u_m$ over $\mathfrak{F}$ and a multiplication table $u_i u_j = \gamma_{ijk} u_k$ for algebraic numbers $\gamma_{ijk}$, where $m = n^2$. If $\delta \neq 0$ is in $\mathfrak{F}$, we let $\mathfrak{F}_0$ be the field obtained by adjoining the $\gamma_{ijk}$ and $\delta$ to $\mathfrak{R}$. Then $\mathfrak{F} = (\mathfrak{F}_0)_\mathfrak{R}$, where $\mathfrak{F}_0 = u_1\mathfrak{F}_0 + \cdots + u_m\mathfrak{F}_0$ is a central division algebra of degree $n$ over $\mathfrak{F}_0$. The field $\mathfrak{F}_0$ is a finite algebraic extension of $\mathfrak{R}$, and so $\delta = \Delta(x)$, where $x$ is in $\mathfrak{D}_0$. But $\Delta(x)$ is the norm form of $\mathfrak{D}$, and our result is proved.

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1 An exposition of this theory is given as chap. IX of the author's Structure of Algebras.


3 This is a consequence of Theorems 9.31 and 9.23.

4 See Theorem 9.21.

where $c, \mu$ are positive constants and $b < 2$. Since $\int_0^1 k(u) \, du = \infty$ (infinite cross-section), with probability 1 infinitely many photons are produced by an electron in each $t$-interval in which it has positive energy.

2. The Straggling* and Generating Functions When $\beta = 0$.—Let $\varepsilon_0(t)$ be the energy at $t$ of an electron which had initially energy 1, in the case $\beta = 0$. Putting $X_0(t) = -\log \varepsilon_0(t)$, it can be seen that $X_0(t)$ has for $t > 0$ a distribution of the infinitely divisible type with a probability density $h_1(x)$ whose characteristic function is, for $x > 0$,

$$\int_0^\infty h_1(x)e^{ixz} \, dx = \exp \left\{ t \int_0^\infty (e^{izu} - 1)k(1 - e^{-u})e^{-u} \, du \right\}. \quad (2.1)$$

The case $k(u) = -\mu/\log (1 - u)$, where $h_1(x)$ is then $x^{\mu-1}e^{-x}/\Gamma(\mu t)$, was given by Bethe and Heitler.\(^5\)

Let $N(\varepsilon, t)$ be the total number of electrons at $t$ whose energies are greater than $\varepsilon$, and let $f_1(\varepsilon, t)$ and $f_2(\varepsilon, t)$ be the generating functions, $\sum \tau \rho(N(\varepsilon, t) = \tau)\varepsilon^\tau$, if initially (i.e., at $t = 0$) there is, respectively, one photon or one electron of energy 1. To derive equations for $f_1$ and $f_2$, the usual method of regeneration points must be modified, since there is no first photon emission. Since differential equations for $f_1$ and $f_2$ have not previously been justified in the present case of infinite cross-sections, we mention that differentiability in $\varepsilon$ and $t$ of $f_1$ and $f_2$ can be established by direct consideration of the random process, and we get equation (2.3) below; the more simply derived companion equation (2.2) is also given:

$$\frac{\partial f_1(s, \varepsilon, t)}{\partial t} = -\lambda f_1(s, \varepsilon, t) + \lambda \int_0^1 f_2 \left( s, \frac{\varepsilon}{u'}, t \right) f_1 \left( s, \frac{\varepsilon}{1 - u'}, t \right) q(u) \, du, \quad (2.2)$$

$$\frac{\partial f_2(s, \varepsilon, t)}{\partial t} = \int_0^1 \left\{ f_1 \left( s, \frac{\varepsilon}{u'}, t \right) f_2 \left( s, \frac{\varepsilon}{1 - u'}, t \right) - f_2(s, \varepsilon, t) \right\} k(u) \, du, \quad (2.3)$$

with $f_1(s, \varepsilon, 0) = f_1(s, 1, t) = f_2(s, 1, t) = 1, t > 0$, and $f_2(s, \varepsilon, 0) = s$ for $\varepsilon < 1$. Uniqueness of the solution among a wide class of functions follows from the fact that the factorial moments (derivatives of the $f_i$ with respect to $s$ at $s = 1$) have Mellin transforms on $\varepsilon$ which satisfy linear differential equations.

In the case of a finite total cross-section ($\int_0^1 k(u) \, du < \infty$), integral equations were given for the characteristic functional of $N(\varepsilon, t)$ by Bartlett and Kendall* and for the generating functions $f_1$ and $f_2$ by Jánossy* and others. Integral equations can be obtained in the present case using a method similar to the one employed by Moyal* to get his equation (2.22).

3. The Expectation Process When $\beta = 0$.—Define a vector Markov process $(I(t), \xi(t)), t \geq 0$, with $I = 1$ or 2 and $\xi > 0$, by the following scheme (let $K = \int_0^1 uk(u) \, du$). $I(t)$ is itself a two-state temporally homogeneous Markov process (we can consider $t$ as a "time" rather than a "thickness" parameter) with respective probabilities $\lambda \, dt$ and $K \, dt$ for the transitions $1 \to 2$ and $2 \to 1$ in $dt$. Whenever the transition $1 \to 2$ occurs, $\xi(t)$ is multiplied by $u$, $0 < u < 1$, with probability density $2uq(u)$, and whenever $2 \to 1$ occurs, $\xi(t)$ is multiplied by $u$ with probability density $uk(u)/K$;
furthermore, if $I(t) \equiv 2$ for $t_1 \leq t \leq t_2$, then $\xi(t)/\xi(t_0)$ decreases, in the interval $t_1 \leq t \leq t_2$, in the manner of the straggling process of Section 2, except that the function $k(u)$ is replaced by $(1 - u)k(u); \xi(t)$ remains constant in any $t$-interval in which $I(t) \equiv 1$. We may think of $I(t)$ and $\xi(t)$ as describing the condition of a single particle which can be sometimes a photon ($I = 1$) and sometimes an electron ($I = 2$) and which has an energy $\xi(t)$. We call $(I(t), \xi(t))$ the "expectation process," for reasons to be seen below.

If we take $\xi(0) = 1$ and $I(0) = i$, $i = 1$ or 2, then if $0 < \epsilon < 1$, $(I(t), \xi(t))$ has for $t > 0$ a probability density $p_{ij}(\epsilon, t) \, d\epsilon$ for the probability that $I(t) = j$ and $\epsilon < \xi(t) < \epsilon + d\epsilon$. Let $m_{ij}(\epsilon, t) \, d\epsilon$ be the expected number, for the cascade process with $\beta = 0$, of particles of type $j$ at $t$ with energies between $\epsilon$ and $\epsilon + d\epsilon$, starting with one particle of type $i$ and energy 1; $i, j = 1$ for photons and 2 for electrons. (These expectation densities are easily seen to exist for $t > 0, 0 < \epsilon < 1$.)

**Theorem.** We have $p_{ij}(\epsilon, t) = m_{ij}(\epsilon, t)$. This can be seen by comparing differential equations in $t$ for $p_{ij}$ with those for $m_{ij}$; the latter can be obtained from equations (2.2) and (2.3) and the corresponding equations for the number of photons. Similarly related Markov processes can, of course, be found for other branching processes.

Since for the expectation process there is, with probability 1, a smallest $t$ where $I$ changes, we have a natural means of using the regeneration-point method to get integral equations for the $p_{ij}$ and hence for the $m_{ij}$; the equations obtained in this way are different in form from those mentioned at the end of Section 2. Here we merely mention that these integral equations are convenient for proving differentiability properties in $\epsilon$ of the $m_{ij}$ or $p_{ij}$. The correspondence between the two processes might be useful in artificial sampling (Monte Carlo) experiments; using the expectation process to determine $p$ and hence $m$, one would deal with less complicated histories than if the original process were sampled directly.

4. **Steady States and Limits for the Case $\beta = 0$**.—The process $(I(t), \xi(t))$ has the stationary "distribution" which assigns the weights $K(a_2 - a_1)$ and $\lambda(a_2 - a_1)$ to the events $\{I(t) = 1, a_1 < -\log \xi(t) < a_2\}$ and $\{I(t) = 2, a_1 < -\log \xi(t) < a_2\}$. Hence the original cascade has the stationary densities $K \, d\epsilon/\epsilon^2$ and $\lambda \, d\epsilon/\epsilon^2$ for the expected number of photons or electrons in the energy range $\epsilon, \epsilon + d\epsilon$. For particles of a single type, a corresponding result was given by Jánossy.11

Next let $z(t)$ be the total energy at $t$ in all electrons, for the cascade process with $\beta = 0$. Since $E[z(t)] = P[I(t) = 2]$, and since $P[I(t) = 2]$ has from the theory of Markov processes the limit $\lambda/(\lambda + K)$ as $t \to \infty$, we have (the symbol $E$ denotes expected value)

$$
\lim_{t \to \infty} E[z(t)] = \frac{\lambda}{\lambda + K}, \quad (4.1)
$$

a result given by Bhabha and Chakrabarty.2 We can strengthen equation (4.1) to get convergence in probability also.12 In fact,

$$
\lim_{t \to \infty} E\left[z(t) - \frac{\lambda}{\lambda + K}\right]^2 = 0. \quad (4.2)
$$

5. **Cascades with $\beta > 0$**.—Let $\epsilon_0(t)$ be a random function defined as in Section 2; for any $\beta > 0$, define

$$
\epsilon_0(t) = \frac{\lambda}{\lambda + K}.
$$
\[
\epsilon_1(t) = \text{Max}\left\{0, \epsilon_0(t) \left[1 - \beta \int_0^t \frac{ds}{\epsilon_0(s)}\right]\right\}.
\]

(5.1)

This formula, which has apparently not been noticed in the literature, gives a representation for the straggling when \(\beta\) is the collision factor. It may be of use in sampling experiments, since it gives a convenient way of constructing sample \(\epsilon_1\)-functions corresponding to various values of \(\beta\), when a single sample \(\epsilon_0\)-function is available.\(^{14}\)

Now let \(N(t)\) be the total number of electrons of all energies at \(t\), electrons whose energies have reached 0 not being counted. It must be recalled that we are using the usual simplifying assumption that the functions \(q\) and \(k\) of Section 1 do not depend on energy, so that no lower limit is assumed for the energy of an electron, whereas in actuality an electron has a certain minimum (rest) energy. However, even for this simplified model Bhabha and Chakrabarty\(^2\) obtained from the moment equations the result (deducible also from energy considerations) that \(E[N(t)]\) is finite when \(\beta > 0\). We can strengthen this to the result that \(E[N^2(t)]\) is also finite when \(\beta > 0\). However, a direct analysis of the successive generations of the process shows that if there is one initial electron, then, with probability 1 \(N(t)\) reaches \(\infty\) in every \(t\)-interval. Although this result must be regarded as a peculiarity of the mathematical model (the so-called “Approximation B”), it suggests that in reality the distribution of the maximum of \(N(t)\) over a \(t\)-interval may differ markedly from the distribution for fixed \(t\).


\(^{4}\) “Straggling” is the name applied to the random decrease in energy of an electron.


\(^{9}\) Both \(p\) and \(m\) have versions which are continuous in \(\epsilon\); taking these, the equality holds for all, not just almost all, \(\epsilon\).


\(^{12}\) The random process \(e^{(K+\lambda)t}|z(t) - \lambda/(K + \lambda)|\) is a martingale but possibly not a convergent one; at least its mean square is unbounded at \(t \to \infty\). Equation (4.2) can be shown by considering a functional equation for the characteristic function of \(z(t)\).

\(^{13}\) The \(\epsilon_0\)-process is defined in such a way that with probability 1 the sample functions are positive nonincreasing step functions (although with infinitely many jumps in each \(t\)-interval).