ON A WEDDERBURN-ARTIN STRUCTURE THEORY OF A POTENT SEMIRING*

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1. Introduction.—In a recent paper¹ we proved that if $S$ is a potent semiring in which each two-sided ideal contains a minimal right ideal and a minimal left ideal of $S$, then any right ideal $R \neq (0)$ contains a multiplicative idempotent. The existence of this idempotent now enables us to obtain for a potent semiring a theory analogous to the Wedderburn-Artin structure theory of a semisimple ring with minimum condition.²

In this paper we prove that if $S$ is a potent semiring with identity in which each two-sided ideal contains a minimum right ideal and a minimum left ideal and $S$ is a strong direct sum of minimal right ideals, then $S$ is a strong direct sum of semirings isomorphic to matrix semirings over division semirings.

2. Strong Direct Sum.—We shall assume that the semiring $S$ possess a zero $0$, in the sense of Vandiver and Weaver,³ $0 + s = s = 0, 0 \cdot s = s \cdot 0 = 0$, for all $s$ in $S$. For the sake of completeness, we repeat the definition of a potent semiring given elsewhere.¹

Definition 1: A semiring $S$ is said to be potent if it contains no nonzero nilpotent right ideals and no nonzero nilpotent left ideals.

Definition 2: A semiring $S$ is said to be simple if it contains no proper two-sided ideals.

Lemma 1. If $S$ is a potent simple semiring and $R \neq (0)$ a minimal right ideal of $S$, then the mappings $\bar{p}(r) = \left( \frac{r}{p} \right), r \in R, p \in S$, and $pR \neq (0)$, form a division semiring.

Proof: Let $\Delta$ be the set of all minimal left ideals $\neq (0)$ of $S$. Then a former lemma¹ states that $S = \sum_{L \in \Delta} L$ and hence $S \rho = \sum_{L \in \Delta} L \rho$. We note that $L \rho$ is either in $\Delta$ or $L \rho = (0)$. If $S \rho R = (0)$, then $S \rho R = SR = (0)$ and $RR = (0)$, which contradicts the potency of $S$. Therefore, $S \rho R \neq (0)$. Since $S \rho = \sum_{L \in \Delta} L$, there exists an $L \in \Delta$ such that $L \rho R \neq (0)$. We set $L_1 = L \rho$. Since $L_1 R \neq (0)$, then $L_1 \neq (0)$ and $L_1 \in \Delta$. Since $L_1 R \neq (0)$, a former lemma¹ states that $L_1 \cap R \ni e = e^2 \neq (0)$. Hence $eR \neq (0)$ and $eR = R$, $er = r$ for $r \in R$. Since $e \in L_1 \cap R \subseteq L_1 \subseteq \mathcal{S} \rho$, then $e = sp, s \in S$. Therefore, $r = er = spR = \bar{p}(r)$, where $\bar{p} = \frac{r}{sp}$ and $\bar{p} = \bar{q}$.
$p$ is a one-to-one mapping of $R$ onto itself, which implies that $s = p^{-1}$ and $p(r) = pr = pe(r)$, where $p \in R$.

**Definition 3:** A semiring $S$ is said to be a weak direct sum $S_1 \oplus S_2$, where $S_1$, $S_2$ are subsemirings with zero, if $S = S_1 + S_2$ and $S_1 \cap S_2 = \{0\}$.

**Definition 4:** A semiring $S$ is said to be a strong direct sum $S_1 + S_2 + \ldots + S_n$, $i = 1, 2, \ldots, n$, if $S = S_1 + S_2 + \ldots + S_n$ and $S_1 \cap S_2 = \ldots = S_{i-1} \cap S_i = \{0\}$ for $s = s_1 + s_2 + \ldots + s_n$, where $s_1, s_2, \ldots, s_n$ are elements of $S_1, S_2, \ldots, S_n$, respectively.

**Lemma 2.** If $S$ is a semiring with identity and $e_1 + e_2$, $e_1^2 = e_1$, then $S$ is a semiring with zero.

**Proof:** Let $S$ be a semiring with identity and $e_1 + e_2$, $e_1^2 = e_1$, then $S$ is a semiring with zero.

**Lemma 3.** There exist left isomorphisms $\sigma_i$ of $R_1$ onto $R_i$ ($\sigma_i = \text{identity isomorphisms of } R_i$).

**Proof:** If $R_iR_i = (0)$, the right annihilator ideal of $R_i$ is a two-sided ideal $\neq (0)$. Hence it is $S$ and $R_iS = (0)$, $R_iR_i = (0)$, which contradicts the potency of $S$. Therefore, $R_iR_i \neq (0)$, and there exists $x \in R_i$ such that $(0) \subset xR_i \subseteq R_i$. Since $xR_i$ is a right ideal and $R_i$ is a minimal right ideal $\neq (0)$, then $xR_i = R_i$. Hence $xR_i$ induces a left homomorphism $\sigma_i$ of $R_1$ onto $R_i$. There exists $y_i$ such that $y_iR_i = R_i$. Therefore, $y_iR_i = y_iR_i = R_i$. Lemma 1 implies that $(y_iR_i)$ induces an isomorphism on $R_i$. Hence $xR_i$ induces an isomorphism of $R_1$ onto $R_i$.

**Lemma 4.** Let $K$ be the semiring of all left homomorphisms of $R_1$ into $R_1$; then $S_1 \cong M_n \times K$ (the semiring of matrices over $K$).

**Proof:** Let $e_\alpha$ be the left homomorphisms of $S$ into $S$, that is defined by $e_\alpha\left(\sum_{j=1}^{n} \sigma_j x_j \right) = \sigma_\alpha x_j$, $x_j \in R_1$. We show that $e_\alpha \cdot e_{\tau_\alpha} = \delta_{\alpha\tau} e_\alpha$ and $\sum_{i=1}^{n} e_{\tau_i} = 1$ implies that $S_1 \cong M_n \times K$, where $K$ is the semiring of all left homomorphisms of $S$ into $S$ permutable with $e_{\tau_i}, i, k = 1, 2, \ldots, n$.

Let $x \in S_1$ and $x \alpha = \sum_{j=1}^{n} e_{\alpha j} x_{\alpha j}$; then $x_{\alpha} e_{\tau_i} = \sum_{j=1}^{n} e_{\alpha j} x_{\alpha j} e_{\tau_i} = \tau_i x_{\alpha j}$. Similarly, $e_{\alpha j} x_{\alpha j} = e_{\alpha j} x_{\alpha j}$. This implies that $x_{\alpha} \in K$. Now $\sum_{x} e_{\alpha j} = \sum_{\gamma} e_{\gamma j} x_{\alpha j} = \sum_{\gamma} e_{\gamma j} x_{\alpha j}$. Conversely, if $y_\alpha \in K$, $x = \sum_{\alpha} y_{\alpha} e_{\alpha j}$, then $x = \sum_{\gamma} e_{\gamma j} y_{\alpha} e_{\alpha j} = y_\alpha \sum_{\gamma} e_{\gamma j} = y_\alpha$. Hence $S_1 \cong M_n \times K$.

$K$ is the semiring of left homomorphisms of $S$ into $S$ permutable with $e_{\tau_i}, i, k = 1, 2, \ldots, n$. Let $k \in K$, and set $\tau(k)(\alpha) = k(\alpha)$. $\tau$ defines a homomorphism of $K$ into $K$. $k \rightarrow \tau(k)$ is a homomorphism of $K$. Let $k \in K$ and $k = \sum_{i} e_{\alpha i} k e_{\alpha i}$; then
\(ke_{\pi} = e_{\pi}k\) and \(k \in K\). Now \(\tau(k)(r_{\lambda}) = \sum e_{\mu}ke_{\mu}(r_{\lambda}) = e_{\mu}k(r_{\lambda}) = k(r_{\lambda})\). Hence \(\tau(k) = k\), and \(\tau\) is an isomorphism of \(K\) onto \(K\). Hence \(S_{i} \cong M_{n} \times K\).

**Lemma 5.** \(K\) is a division semiring.

Proof: If \(k \in K\), \(k \neq 0\) and \(R_{1} = eS_{i}, e^{2} = e\), then \(k = (k(e))_{e}\), when applied to \(R_{1}\); for \(k(ex) = k(eex) = k(e)ex\). Lemma 1 states that for each \(\rho \in S\) for which \(\rho R_{1} \subseteq R_{1}, \rho R_{1} \neq (0)\), there exists \(\rho' \in R\) such that \(\left(\begin{array}{c} r_{1} \\ \rho'r_{1}\end{array}\right)\) is the inverse of \(\left(\begin{array}{c} r_{1} \\ \rho r_{1}\end{array}\right)\).

Actually \(K\) is formed already by the left multiplications of \(R_{1}\) with elements of \(R_{1}\), since \(\rho\) and \(\rho'\) induce the same mapping of \(R_{1}\) into \(R_{1}\).

If \(S\) possesses an identity, then \(S\) is isomorphic to \(S_{i}\). Thus Lemmas 3–5 yield the following structure theorems:

**Theorem 1.** If \(S\) is a potent simple semiring with identity and \(S\) is a strong direct sum of minimal right ideals, then \(S\) is isomorphic to a semiring of matrices over a division semiring.

**Theorem 2.** If \(S\) is a potent semiring with identity, in which each two-sided ideal contains a minimal right ideal and a minimal left ideal, and \(S\) is a strong direct sum of minimal right ideals, then \(S\) is a strong direct sum of semirings which are isomorphic to semirings of matrices over division semirings.

Proof: Since \(S = R_{1} + R_{2} + \ldots + R_{n}, R_{i}, \ i = 1, 2, \ldots, n\), minimal right ideals, we collect all summands which are operator isomorphic. These operator-isomorphic summands form a minimal two-sided ideal. \(S\) is a strong direct sum of these minimal two-sided ideals, which are simple with identity. Their structure is known by Theorem 1.

We conclude this paper with an example of a matrix semiring over a division semiring, in which the modular identity and strong decomposition do not hold. Let \(M_{n} \times S = S_{n}\) be the matrix semiring of order \(n\), over a division semiring \(S\). Let \(1 < i \leq n, E_{i} = e_{11} + e_{22} + \ldots + e_{ii}\); then \(e_{ij}S_{n}\) is a minimal right ideal, and \(E_{i+i}S_{n} = E_{i}S_{n} + e_{i+i} + e_{i}S_{n}\). If \(S\) is not a division ring, then \(E_{i}S_{n}\) is not maximal in \(E_{i+i}S_{n}\), because, in the case where \(S\) is a division semiring for any \(\xi \neq 0\) of \(S\), we have \(S + \xi + S \subset S\). Hence for any element of the form \(p = e_{i+i} + y, 0 \neq y \in E_{i}S_{n}\), we find that \(E_{i}S_{n} \subset E_{i}S_{n} + pS_{n} \subset E_{i+i}S_{n}\). If \(S\) is the division semiring of non-negative real numbers, then there is the infinite descending chain formed by the intermediate ideals \(E_{i}S_{n} + (e_{i+i} + y + (1/m)e_{i})S_{n}, m = 1, 2, \ldots\). Hence the modularity identity is not valid in \(S_{n}\), and no strong decomposability exists either.

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