NON-ADDITIVE FUNCTORS, THEIR DERIVED FUNCTORS, AND THE SUSPENSION HOMOMORPHISM

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1. Derived Functors of Non-additive Functors.—Let $T$ be a (covariant) functor of modules over a ring $A$ to modules over a ring $A'$. If $T$ is additive its derived functors have been defined by Cartan-Eilenberg. Additivity is used to show that $T$ applied to a chain homotopy again gives a chain homotopy (cf. W. Craig, J. Symb. Logic, 18, 30–32, 1953, and B. H. Neumann, J. London Math. Soc., 12, 125, Theorem (13), 1937).

Using $FD$-complexes instead of chain complexes, we define left derived functors for arbitrary functors $T$.

1.1. Definition.—A projective $FD$-resolution of type $n$ of the module $M$ is an $FD$-module $P$ such that (i) $P_j = 0$ for $j < n$, (ii) $P_j$ is projective for all $j$, (iii) $H_n(P) = M$, $H_j(P) = 0$ for $j \neq n$.

Passing from an $FD$-module to its normal (ized) chain module establishes a 1 to 1 correspondence (up to natural equivalences) between $FD$-modules, $FD$-maps, $FD$-homotopies and chain modules, chain maps, chain homotopies. In particular, it establishes such a correspondence between projective $FD$-resolutions $P$ of type $n$ of $M$ and chain modules $C$ for which $C_j = 0$ for $j < n$ and

$$0 \rightarrow M = H_n(C) \leftarrow C_n \leftarrow C_{n+1} \leftarrow C_{n+2} \leftarrow \ldots \quad (1.2)$$

is an ordinary projective resolution of $M$. From the corresponding properties of ordinary resolutions it follows: Every module has a projective $FD$-resolution of any given type $n$. If $f: M \rightarrow M'$ is a homomorphism and $P, P'$ are projective $FD$- resolutions of type $n$ of $M, M'$ resp., then there exists an $FD$-map $F: P \rightarrow P'$ such that $F_*: H_n(P) \rightarrow H_n(P')$ equals $f$. Moreover, if $F'$ is another such map, then $F, F'$ are $FD$-homotopic (cf. Cartan-Eilenberg, 2, V, 1).
Now let \( T \) be an arbitrary functor from modules to modules. Its prolongation\(^3\) to \( FD \)-modules (defined by applying \( T \) is every dimension and also to the face- and degeneracy operators) preserves homotopy, i.e., \( F \simeq F' \) implies \( TF \simeq TF' \) for all \( FD \)-maps \( F,F' \), even if \( T \) is not additive.\(^3\) Hence it follows, as in Cartan-Eilenberg\(^2\) (V, 3), that \( H_q(TP) \) depends only on \( M \) and \( n \) (not on \( P \)), and \((TF)_* : H_q(TP) \to H_q(TP') \) depends only on \( f \) and \( n \). We denote \( H_q(TP) \), \((TF)_* \) by \( L_{q}^{(n)}TM \), \( L_{q}^{(n)}Tf \), resp., and call \( L_{q}^{(n)}T \) the \( q \)th left derived functor of \( T \) of type \( n \).

If \( T \) is additive, \( L_{q}^{(n)}T \) coincides with the left derived functor \( L_{q}^{(n)}T \) as defined in Cartan-Eilenberg.\(^2\) For arbitrary \( T \) we shall exhibit a natural transformation \( \sigma : L_{q}^{(n)}T \to L_{q}^{(n+1)}T \), the suspension (cf. (2.4)). \( \sigma \) is an isomorphism for additive \( T \), but not in general.

1.3. Examples.—Let \( T,T' \) be the functors from abelian groups to \( \Lambda \)-modules defined by \( TM = N \otimes Z(M) \), \( T'M = N \otimes SA(M) \), where \( N \) is a fixed \( \Lambda \)-module, \( Z(M) \) is the group ring, and \( SA(M) \) is the symmetric algebra of the abelian group \( M \). Then \( L_{q}^{(n)}TM \) for all \( n \), and \( L_{q}^{(n)}T'M \) for \( n > 0 \) are the Eilenberg-Mac Lane modules \( H_q(M,n;N) \).\(^4\) We do not know how to answer the interesting question as to the nature of the derived functors of the \( \Gamma \)-functor of Eilenberg-Mac Lane.\(^6\)

2. The Suspension.—Define the cone over an \( FD \) module \( K \) to be the \( FD \)-module \( CK \) given by

\[
\begin{align*}
(CK)_q &= K_q + K_{q-1} + K_{q-2} + \ldots + K_0, \\
\partial_i(a_q, a_{q-1}, \ldots, a_0) &= (\partial_0 a_q, \partial_1 a_{q-1}, \ldots, \partial_i a_{q-i+1}, \partial_{i-1} a_{q-i} + a_{q-i-1}, a_{q-i-2}, \ldots, a_0), \quad i < q, \\
\varepsilon_i(a_q, a_{q-1}, \ldots, a_0) &= (\varepsilon_0 a_q, \varepsilon_1 a_{q-1}, \ldots, \varepsilon_i a_i).
\end{align*}
\]

(2.1)

where \( \partial_i \) and \( \varepsilon_i \) are the face- and degeneracy operators. There is a natural injection \( \iota : K \to CK \), and the \( FD \)-module \( SK = CK/iK \) is called the suspension (cf. (2.4)). The exact sequence

\[
0 \to K \to CK \to SK \to 0
\]

(2.2)

splits if we disregard the \( FD \)-structures, i.e., \((CK)_q \cong K_q + (SK)_q \) for all \( q \). \( CK \) is of the same homotopy type as the trivial complex 0 (notation: \( CK \equiv 0 \)), hence \( H_q(K) \cong H_{q+1}(SK) \). If \( P \) is a projective resolution of type \( n \), then \( SP \) is a projective resolution of type \( n + 1 \) of the same module.

Let \( T \) be a functor from modules to modules prolonged to \( FD \)-modules as above, and assume \( T(0) = 0 \). Then \((Tp)(Ti) = T(pi) = 0 \), i.e., \( im(Ti) \subset ker(Tp) \), and we have \((Ti)_* : H_q(TK) \to H_q(\ker(Tp)) \). Since (2.2) splits, \( Tp \) is an epimorphism, hence \( TSK \cong TCK/\ker(Tp) \). The boundary homomorphism \( \Delta : H_{q+1}(TSK) \to H_q(\ker(Tp)) \) is an isomorphism because \( CK \equiv 0 \) implies \( TCK \equiv T(0) = 0 \) (T preserves homotopy). So we can form the suspension homomorphism,

\[
\sigma = \Delta^{-1}(Ti)_* : H_q(TK) \to H_{q+1}(TSK).
\]

(2.3)

In particular, if \( K = P \) is a projective \( FD \)-resolution of type \( n \) of \( M \), this gives a natural transformation,

\[
\sigma_q^{(n)}(M) : L_q^{(n)}TM \to L_{q+1}^{(n+1)}TM.
\]

(2.4)

If \( T \) does not satisfy \( T(0) = 0 \), then \( TM = T'M + M_0 \), where \( T'(0) = 0 \), and
$M_0$ is a fixed module. In order to apply our constructions, replace $T$ by the functor $T'$.

3. Properties of the Suspension.—Eilenberg and Mac Lane\(^6\) defined the $k$th cross-effect functor of a functor $T$ with $T(0) = 0$. These functors, also denoted by $T$, satisfy

$$T(M_1 + M_2 + \ldots + M_r) = \sum_{i \leq u < i_2 \ldots < u_{k-1} \leq i} T(M_i \mid M_{i_1} \mid \ldots \mid M_{i_k}).$$

(3.1)

Letting all variables in the $k$th cross-effect be equal gives a functor in one variable,

$$T_{[k]}M = T(M \mid M \mid \ldots \mid M) \quad (k \text{ variables}).$$

(3.2)

Let $\sum_k M$ denote the direct sum of $k$ copies of $M$. Define $\alpha'_i': \sum_k M \to \sum_{k-1} M$ and $\beta'_i': \sum_{k-2} M \to \sum_{k-1} M$ for $0 < i < k$ by

$$\alpha'_i'(m_1, m_2, \ldots, m_k) = (m_1, \ldots, m_{i-1}, m_i + m_{i+1}, m_{i+2}, \ldots, m_k) \quad \beta'_i'(m_1, m_2, \ldots, m_{k-1}) = m_1, \ldots, m_{i-1}, m_i, m_{i+1}, \ldots, m_{k-1}).$$

(3.3)

By restriction and projection (cf. 3.1) we obtain from $T\alpha'_i$ and $T\beta'_i$ the natural transformations

$$\alpha_i': T_{[i]}M \to T_{[i-1]}M \quad 0 < i < k.$$  

$$\beta_i: T_{[k-1]}M \to T_{[k]}M \quad 0 < i < k.$$  

(3.4)

$\alpha_i$ and $\beta_i$ are prolonged to FD-modules by applying them in every dimension.

Let $K$ be an FD-module. The elements in the image of $\alpha_i: H(T(K[K])) \to H(TK)$ resp. in the kernel of $\beta_i: H(TK) \to H(T(K[K]))$ are called decomposable resp. primitive.  

3.5 Proposition. The decomposable elements of $H(TK)$ lie in the kernel of the suspension $\sigma: H(TK) \to H(TSK)$. The image of $\sigma$ consists of primitive elements of $H(TSK)$.

To obtain further results on the suspension (in particular a partial inverse of 3.5: cf. 3.9) we consider the sequence

$$TK = T_{[1]}K \leftarrow T_{[2]}K \leftarrow \ldots \leftarrow T_{[k]}K \leftarrow \ldots,$$

(3.6)

where $d = d_k = \sum_{j=1}^k (-1)^j \alpha_j$. Then $dd = 0$, and $\pm d$, together with the "internal" differential $d = \sum (-1)^j \partial_j$ of $T_{[1]}K$, turns $\mathfrak{T}K = \sum T_{[1]}T_{[2]}K$ into a double complex. If $T$ stands for "group ring" or "symmetric algebra" (cf. 1.3) $\mathfrak{T}K$ is very much like the bar construction.  

3.7. Theorem. There is a natural isomorphism $H_{*+1}(TSK) \cong H_*(\mathfrak{T}K)$ which transforms the suspension $\sigma: H_*(TK) \to H_{*+1}(TSK)$ into the map $\mathfrak{i}_*: H_*(TK) \to H_*(\mathfrak{T}K)$ induced by the inclusion $i*: TK \to \mathfrak{T}K$. (H($\mathfrak{T}K$) is the homology of the single complex associated with the double complex $\mathfrak{T}K$; cf. Cartan-Eilenberg,\(^2\) IV, 4.)

There are two spectral sequences associated with the double complex $\mathfrak{T}K$ (cf. Cartan-Eilenberg,\(^2\) XV, 6). One of them can be used to prove 3.7, the other gives

3.8. Corollary. There is a spectral sequence $E'$ such that $E^1$ together with its differential $d^1$ is the (single) complex

$$H(TK) \leftarrow H(T_{[1]}K) \leftarrow \ldots \leftarrow H(T_{[k]}K) \leftarrow \ldots.$$
and $E^m$ is the graded module associated with a certain filtration of $H(TSK)$.

It follows from a generalization of the Eilenberg-Zilber theorem by P. Cartier that $H_q(T_{[1]}K) = 0$ for $q < k_0$ if $(A_n)$: $K_q = 0$ for $q < n$. Using Dold, Section 3, one can replace the hypothesis $(A_n)$ by $(B_n)$: $K$ is a projective FD-module over a hereditary ring and $H_q(K) = 0$ for $q < n$. By standard arguments on spectral sequences one then obtains from 3.8

3.9. Corollary. If $K$ satisfies $(A_n)$ or $(B_n)$ then there is an exact sequence

$$H_q(T_{[1]}K) \rightarrow H_q(TK) \rightarrow H_{q+1}(TSK) \rightarrow H_{q-1}(T_{[2]}K) \rightarrow H_{q-1}(TK) \quad (3.10)$$

for $q \leq 3n$; in particular, $\sigma$: $H_q(TK) \rightarrow H_{q+1}(TSK)$ is an isomorphism for $q < 2n$ and an epimorphism for $q = 2n$.

It can be shown that the homomorphism $\beta$ in (3.10) is the composition of $\delta_{1*}$:

$$H_{q+1}(TSK) \rightarrow H_{q+1}(T_{[1]}SK)$$

and an isomorphism $H_{q+1}(T_{[1]}SK) \cong H_{q-1}(T_{[2]}K)$.

If $T$ is a quadratic functor, i.e., $T(M_1)\langle M_2, M_3 \rangle = 0$ for all $M_1, M_2, M_3$, there is an exact sequence (3.10) without any assumption on $K$ and without restriction on $q$.

All results on the suspension have obvious applications to the left derived functors of $T$. Corollary 3.9 gives, in particular,

$$\sigma \colon L^q(\cdot)TM \cong L^{q+1}_{q+1}TM \quad \text{for} \quad q < 2n. \quad (3.11)$$

There is a "dual" to the construction (3.6) which resembles the cobar construction. Together with the details about the preceding results, it will appear elsewhere.


DIFFERENTIABLE MAPPINGS IN THE SCHOENFLIES PROBLEM

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1. Introduction.—The recent remarkable contributions of Mazur, of Princeton University, to the topological theory of the Schoenflies problem lead naturally to questions of fundamental importance in the theory of differentiable mappings. In particular, if the hypothesis of the Schoenflies Theorem is stated in terms of regular differentiable mappings of class $C^m$, $m > 0$, can the conclusion be stated in terms of regular differentiable mappings of the same class? This paper presents an outline of an answer to this question. It shows how to develop a differential theory corresponding to Mazur's process.1

An abstract $r$-manifold $\Sigma_r$, $r > 1$ of class $C^m$ is understood in the usual sense.5,3