$v_i^{(0)}(t)$, we obtain a system of linear equations whose solution is clearly $x_i = x_i^{(0)}$.

It follows that a set of forcing functions which minimize $G$ subject to the linear equations of (2.1), together with the original constraints, yields a value of $G$ which is at most $G(x_1^{(0)}(T), \ldots, x_N^{(0)}(T))$. The general result follows inductively.

This monotonicity is not surprising, since we are using the technique of approximation in policy space.\(^1\)

3. **Successive Approximation and the Hitchcock-Koopmans Problem.**—Let us now turn to the second problem described in section 1. As a first approximation, let $x_i$ be a set of values satisfying the constraints in (1.4). To obtain a second approximation, we fix the quantities sent out from the sources $i = 3$ to $i = N$, and determine the allocations from the first two sources so as to minimize the cost of supplying the remaining demand. This problem can be resolved in terms of sequences of functions of one variable.\(^6\)

To obtain a third approximation, we fix the allocations from the first source and the sources $i = 4$ to $i = N$, and determine the allocation from the second and third sources so as to minimize the cost of supplying the remaining demands.

Continuing in this fashion, we obtain a sequence of problems, each of whose solutions depends upon a sequence of functions of one variable. As above, it is easy to see that the sequence of costs obtained in this way is monotone decreasing. Once again, interesting questions arise concerning convergence which we do not enter into here.

Similar techniques can be applied to other classes of combinatorial problems as will be shown elsewhere.


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**ON THE THEOREM OF BERTINI FOR LOCAL DOMAINS***

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1. The well-known Theorem of Bertini on reducible linear systems of divisors in an algebraic variety asserts that any such system, assumed to be free from fixed components, is composite with a pencil. A somewhat special form of this theorem, which is actually a special case of it, states that if a linear system of divisors is obtained from a rational transformation of the variety onto a projective
space of dimension >1, then a generic element of the system, apart from a possible fixed component, is absolutely irreducible. In a recent discussion on local algebraic geometry, Professor S. Abhyankar raised the question whether this Theorem of Bertini, at least in the special form, also holds for local varieties, with suitable modifications. In this note we shall answer this question in the affirmative by formulating and proving such a theorem for an arbitrary complete local domain. The theorem we shall prove here can be stated as follows:

**Theorem of Bertini** (for local domains). Let \((\mathcal{R}, p)\) be a complete local domain of dimension \(t > 2\), let \((\mathcal{S}, q)\) be a complete regular local domain of dimension \(t\) contained in \(\mathcal{R}\) such that \(\mathcal{R}\) is a finite module over \(\mathcal{S}\) and is separable over \(\mathcal{S}\), and let \(x_1, x_2, x_3\) be three elements in a minimal basis for \(q\); we set \(u(c) = (x_1 + cx_2)/x_3\), where \(c\) is an element in \(\mathcal{S}\). Then, for sufficiently general \(c\), the ideal \(\mathcal{R}[u(c)]p\) is prime in \(\mathcal{R}[u(c)]\) and the quotient ring \(\mathcal{R}_u(c)\) of \(\mathcal{R}[u(c)]\) with respect to \(\mathcal{R}[u(c)]p\) is analytically irreducible.

We add in explanation that the expression “sufficiently general \(c\)” in the above theorem means that the \(q\)-residue of \(c\) is different from some one given finite set of elements in the residue field \(\mathcal{S}\) of \(\mathcal{S}\).

The above local formulation is somewhat different in spirit from the original version of the Theorem of Bertini, in that it belongs more properly to the relative rather than to the absolute algebraic geometry. This is in a sense natural when one deals with local domains, and we have accordingly replaced the extension to algebraic closure by the extension to the completion of a local domain; furthermore, since the restriction to a local variety diminishes the degree of freedom by 1, the dimensionality condition in the local theorem has naturally to be increased by 1. Nevertheless, our proof of the local theorem will be at least formally similar to the proof given by Zariski\(^1\) of the Theorem of Bertini in the global geometric case. Our proof is based on the following lemma, which may have some interest in itself; the discerning reader will observe that our proof of this lemma, given in the next section, is at least formally similar to the proof of Lemma 5 in the paper of Zariski quoted above, which plays a key role in his proof of the Theorem of Bertini.

**Lemma.** Let \((\mathcal{S}, q)\) be a complete regular local domain of dimension \(t > 2\), and let \(x_1, \ldots, x_t\) be the elements in a minimal basis for \(q\); we set \(u(c) = (x_1 + cx_2)/x_3\), where \(c\) is an element in \(\mathcal{S}\), and denote by \(\mathcal{S}_u(c)\) the quotient ring of \(\mathcal{S}[u(c)]\) with respect to the prime ideal \(\mathcal{S}[u(c)]q\). If a monic polynomial \(F(X)\) in \(\mathcal{S}[X]\) (where \(X\) is an indeterminate) is irreducible over \(\mathcal{S}\), then it is also irreducible over the completion \(\mathcal{S}_u(c)*\) of \(\mathcal{S}_u(c)\) for sufficiently general \(c\).

2. Before proceeding to the proof of our lemma, we shall make a few remarks on the “power-series” development of an element in the complete regular local domain \((\mathcal{S}, q)\). As before, we denote by \(\mathcal{S}\) the residue field of \(\mathcal{S}\), and denote by \(Q\) a system of representatives of \(\mathcal{S}\) in \(\mathcal{S}\); apart from the condition that every element in \(\mathcal{S}\) is the \(q\)-residue of exactly one element in \(Q\), the elements in \(Q\) can be chosen arbitrarily, but, for convenience, we shall assume that \(Q\) contains the elements 0 and 1. Using the properties of a regular local domain, it can easily be shown that every element in \(\mathcal{S}\) can be represented uniquely by a power series in \(x_1, \ldots, x_t\) with coefficients in \(Q\); we shall express this by setting \(\mathcal{S} = Q[[x_1, \ldots, x_t]]\). For a fixed \(1 \leq j < t\), we set \(v_1 = x_j/x_{j+1}, \ldots, v_t = x_{j+1}/x_{j+1}\), and denote the set \((v_1, \ldots, v_t)\). For a fixed \(1 \leq j < t\), we set \(v_1 = x_j/x_{j+1}, \ldots, v_t = x_{j+1}/x_{j+1}\), and denote the set \((v_1, \ldots, v_t)\). For a fixed \(1 \leq j < t\), we set \(v_1 = x_j/x_{j+1}, \ldots, v_t = x_{j+1}/x_{j+1}\), and denote the set \((v_1, \ldots, v_t)\).
whose in an3 and that each polynomial in $\mathbb{Q}(v)$ with coefficients in $\mathbb{Q}$ or canonically exist the element that the uniqueness basis for minimal which the distinct with $c_2$ power-series representation of elements $u_2 x_3, u_2 x_3, x_3, \ldots, x_i$ also form a minimal basis for $p_i$; in the notations introduced above, we then have the relations $\mathbb{E}_{u_i}^* = Q(u_1)(x_3, \ldots, x_i)$, $\mathbb{E}_{u_2}^* = Q(u_2)(x_3, \ldots, x_i)$, and $\mathbb{E}_{u_1, u_2}^* = Q(u_1, u_2)(x_3, \ldots, x_i)$, and we observe that both $\mathbb{E}_{u_i}^*$ and $\mathbb{E}_{u_2}^*$ are here already canonically imbedded in $\mathbb{E}_{u_1, u_2}^*$ as subrings. Let $a$ be any coefficient in $F_1(X)$ or $F_2(X)$, and let $a = \sum a_{n_1} x_3^{n_1} \cdots x_i^{n_i}$ be the power-series development of $a$ with coefficients in $Q(u_1, u_2)$. Since $a$ is contained in $\mathbb{E}_{u_1}^*$, it can be represented by a power series in $u_2 x_3, x_3, \ldots, x_i$ with coefficients in $Q(u_1)$; it then follows from the uniqueness of the power-series representation of $a$ that each $a_{n_1} \ldots n_i$ is a polynomial in $u_1$ of degree at most $n_3$. Similarly, interchanging $u_1$ and $u_2$, we can show that each $a_{n_1} \ldots n_i$ is a polynomial in $u_2$ of degree at most $n_3$. Thus we have shown that each coefficient $a_{n_1} \ldots n_i$ is an element in $Q[u_1, u_2]$ of non-positive $u_1$-excess and non-positive $u_2$-excess. In order to prove our lemma, it is sufficient to show that the $(u_1, u_2)$-excess of each $a_{n_1} \ldots n_i$ is non-positive. For, by a remark made above, the element $a$ would then be contained in $\mathbb{E}$, and since $a$ is any coefficient in $F_1(X)$ or $F_2(X)$, both $F_1(X)$ and $F_2(X)$ would then be elements in $\mathbb{E}[X]$; this would be in contradiction to our assumption that $F(X)$ is irreducible in $S[X]$.

We have the relation $u_2 = du_1 + eu_3$, where $d = (c_3 - c_2)/(c_3 - c_1)$ and $e = (c_2 - c_1)/(c_3 - c_1)$ are units in $\mathbb{E}$; if we substitute for $u_2$ the expression $du_1 + eu_3$ in $a_{n_1} \ldots n_i$, we obtain a polynomial $a_{n_1} \ldots n_i$ in $u_1$ and $u_3$ with coefficients in $S$, whose degree in $u_1$ is equal to the total degree of $a_{n_1} \ldots n_i$ in $u_1$ and $u_2$. The ex-
pression \( a = \sum a_{n_1} \ldots n_l' x_1^{n_1'} \ldots x_l^{n_l'} \) is then a development of \( a \) as a power series in \( x_3, \ldots, x_l \) with coefficients in \( \mathcal{S}[u_1, u_3] \), and if \( x_3^{n_3} \ldots x_l^{n_l} \) is a term of the lowest degree such that the coefficient \( a_{m_1} \ldots m_l' \) has a positive \((u_1, u_3)\)-excess, then it is also a term of the lowest degree such that the coefficient \( a_{m_1} \ldots m_l' \) has a positive \( u_1 \)-excess. We observe now that if \( f(u_1, u_3) \) is any element in \( \mathcal{S}[u_1, u_3] \) and if we represent each coefficient in \( f(u_1, u_3) \) by a power series in \( u_3 x_3, u_3 x_3, x_3, \ldots, x_3 \) with coefficients in \( Q \), then we obtain a development of \( f(u_1, u_3) \) as a power series in \( x_3, \ldots, x_l \) with coefficients in \( \mathcal{S}[u_1, u_3] \), but the coefficient of the term of degree zero is an element in \( Q[u_1, u_3] \); and it is easily seen that the \( u_1 \)-excess of the coefficient of each term in this development is at most equal to the degree of \( f(u_1, u_3) \) in \( u_1 \). If we now, beginning with the terms of the lowest degree, successively substitute in the power series \( a = \sum a_{n_1} \ldots n_l' x_1^{n_1'} \ldots x_l^{n_l'} \) the coefficients of the terms of a given degree by power-series developments of the type just described, we shall obtain a development of \( a \) as a power series \( a = \sum b_{n_1} \ldots n_l x_3^{n_1} \ldots x_l^{n_l} \) in \( x_3, \ldots, x_l \) with coefficients in \( Q[u_1, u_3] \), and it can be easily seen from what we have said above that the \( u_1 \)-excess of \( b_{m_1} \ldots m_l' \) must be positive. However, this is impossible; for the expression \( a = \sum b_{n_1} \ldots n_l x_3^{n_1} \ldots x_l^{n_l} \) is the unique development of \( a \) as a power series in \( x_3, \ldots, x_l \) with coefficients in \( Q[u_1, u_3] \), and by a similar argument as before with \( u_3 \) replacing \( u_2 \), we can show that the \( u_1 \)-excess of each coefficient \( b_{n_1} \ldots n_l \) must be non-positive. This completes the proof of the lemma.

3. Proof of the Theorem of Bertini: Since \( u(c) \) is the quotient of two elements in a system of parameters in \( \mathcal{R} \), it can be easily shown, by a well-known argument, that the \( \mathcal{R}[u(c)]_p \)-residue of \( u(c) \) is a variable over \( \mathcal{R} \), so that \( \mathcal{R}[u(c)]/\mathcal{R}[u(c)]_p \) is a polynomial ring in one variable over \( \mathcal{R} \); this shows that \( \mathcal{R}[u(c)]_p \) is a prime ideal in \( \mathcal{R}[u(c)] \) and that \( \mathcal{R}[u(c)]_p \) contracts to \( \mathcal{S}[u(c)]_o \) in \( \mathcal{S}[u(c)] \). In order to prove our theorem, it is sufficient to show that \( \mathcal{R}\mathcal{S}_{u(c)} \) is an analytically irreducible local domain; for, since \( \mathcal{R}\mathcal{S}_{u(c)} \) contains \( \mathcal{R}[u(c)] \) and is contained in \( \mathcal{R}_u(c) \) and since the maximal prime ideal in \( \mathcal{R}\mathcal{S}_{u(c)} \) must then contract to \( \mathcal{R}[u(c)]_p \) in \( \mathcal{R}[u(c)] \), one sees readily that \( \mathcal{R}_{u(c)} = \mathcal{R}\mathcal{S}_{u(c)} \). Since \( \mathcal{R}\mathcal{S}_{u(c)} \) is a finite module over the local domain \( \mathcal{S}_{u(c)} \), it is a semilocal domain in the sense of Chevalley; it is therefore sufficient to show that the completion \( (\mathcal{R}\mathcal{S}_{u(c)})^* \) of \( \mathcal{R}\mathcal{S}_{u(c)} \) is a local domain. Let \( K \) and \( L \) be the quotient fields of \( \mathcal{R} \) and \( \mathcal{S} \) respectively, and let \( L^* \) be the quotient field of the completion \( \mathcal{S}_{u(c)}^* \) of \( \mathcal{S}_{u(c)} \), which we recall is a regular local ring. According to a result of Chevalley, \( (\mathcal{R}\mathcal{S}_{u(c)})^* \) is a finite module over \( \mathcal{S}_{u(c)}^* \) and no non-zero element in \( \mathcal{S}_{u(c)}^* \) is a zero-divisor in \( (\mathcal{R}\mathcal{S}_{u(c)})^* \); if we denote by \( M \) the quotient ring of \( (\mathcal{R}\mathcal{S}_{u(c)})^* \) with respect to the set of all non-zero elements in \( \mathcal{S}_{u(c)}^* \), then both \( K \) and \( L^* \) are subrings in \( M \) and we have the equation \( M = KL^* \) (\( KL^* \) denotes here the ring generated by \( K \) and \( L^* \)). Moreover, by the same result, every set of linearly independent elements in \( K \) over \( L \) remains such over \( L^* \); it follows from this that \( M \) is isomorphic to the tensor product \( K \times L^* \) over \( L \). Now, since \( K \) is separable over \( L \), there exist a finite number of irreducible monic polynomials \( F_i(X) \) in \( \mathcal{S}[X] \) such that any extension field \( L_i \) of \( L \) is linearly disjoint with respect to \( K \) over \( L \) if and only if all \( F_i(X) \) remain irreducible in \( L_i[X] \). According to our lemma, for sufficiently general \( c \), all \( F_i(X) \) will remain irreducible in \( \mathcal{S}_{u(c)}^*[X] \); since \( \mathcal{S}_{u(c)}^* \) is integrally closed, this implies that all \( F_i(X) \) will remain irreducible in \( L^*[X] \). It follows that \( L^* \) is linearly disjoint with respect to \( K \).
over $L$, and hence the tensor product $K \times L^*$ is an integral domain; this implies of course that $(R \otimes u(c))^*$ is an integral domain. Finally, since $(R \otimes u(c))^*$ is a finite module over the complete local domain $\mathcal{E}_{u(c)}^*$ and has no zero-divisors, it must be a complete local domain.\footnote{This work was partially supported by a research grant of the National Science Foundation.} This completes the proof of our theorem.

We shall conclude this note by a few remarks on the hypothesis of our theorem. We observe that in our formulation of the local Theorem of Bertini, we have introduced the subring $\mathcal{E}$ and a system of regular parameters $x_1, \ldots, x_t$ in $\mathcal{E}$, while in the final assertion of the theorem only the three elements $x_1, x_2, x_3$ enter into the picture. The question naturally arises whether it is necessary at all to introduce the regular local domain $\mathcal{E}$ or even the system of parameters $x_1, \ldots, x_t$ in $\mathcal{E}$. As to the latter, one finds a measure of justification in the fact that, even in the equal-characteristic case where $\mathcal{R}$ contains a coefficient field $k$, an arbitrary set of three analytically independent elements $x_1, x_2, x_3$ may generate in $\mathcal{R}$ an ideal of rank less than 3 and hence cannot be extended into a system of parameters in $\mathcal{R}$. This is a well-known situation where a local ring may contain as a subring a local ring of higher dimension, and one can easily obtain examples to show that the local Theorem of Bertini cannot hold for an arbitrary set of three analytically independent elements without some qualifications. As to the introduction of the regular local domain $\mathcal{E}$, it is evidently not necessary in the equal-characteristic case, since any system of parameters $x_1, \ldots, x_t$ in $\mathcal{R}$ can be considered as the minimal basis of the maximal prime ideal in the regular local domain $k[[x_1, \ldots, x_t]]$, where $k$ is a coefficient field in $\mathcal{R}$. However, in the unequal-characteristic case, we do not know whether every system of parameters in $\mathcal{R}$ can be so imbedded in a regular local domain, and such an imbedding is essential for our present method of proof. The question therefore remains open whether the local Theorem of Bertini holds in general for any three elements in a system of parameters in $\mathcal{R}$.

\footnote{\textit{Ibid.}, § II, Proposition 7.}
\footnote{\textit{Ibid.}, § III, Proposition 8. The proof of this proposition also holds for a local ring in the sense of Krull.}

\section*{THE STRUCTURE OF PRIME RINGS WITH MAXIMUM CONDITIONS}

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1. \textit{Introduction}.—The purpose of this note is to indicate briefly the proof of the following theorem. A detailed account will appear elsewhere.

\textbf{Theorem A.} A ring $R$ has a right and left quotient ring which is a complete matrix ring $M_n(D)$, where $D$ is a division ring, if and only if (a) $R$ is a prime ring; (b) $R$ has