SUPPLEMENT TO A PAPER ENTITLED "A VON STERNECK
ARITHMETICAL FUNCTION AND RESTRICTED PARTITIONS WITH
RESPECT TO A MODULUS"

BY C. A. NICOL AND H. S. VANDIVER*

ILLINOIS INSTITUTE OF TECHNOLOGY, THE UNIVERSITY OF TEXAS

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In our article entitled "A Von Sterneck Arithmetical Function and Restricted
Partitions with Respect to a Modulus" in these Proceedings, 40, 834, n. 3, 1954, we
said: "We hope that other mathematicians will check our conclusion that Von
Sterneck was the first investigator to isolate this number and obtain a few of its
properties." The number referred to in this footnote is as follows:

\[ \Phi(k, n) = \frac{\phi(n)}{\phi(n/(k, n))} \mu \left( \frac{n}{(k, n)} \right), \tag{1} \]

where \( \phi(h) \) is the Euler indicator of \( h \) and \( \mu(h) \) is the Möbius number for \( h \). Also,
\( (k, n) \) is the greatest common divisor of \( k \) and \( n \); \( k \) and \( n \) are integers with \( k \geq 0 \),
\( n > 0 \) and where \( (0, n) = n \).

At a mathematical meeting in Pasadena, California, in June, 1955, L. Schoenfeld
and T. M. Apostol kindly called our attention to the fact that Dedekind, in his
fourth edition of Dirichlet's Vorlesungen über Zahlentheorie (1894), page 369, foot-
note, observed:

... ist \( m = m'P \) eine beliebige positive ganze Zahl, \( P \) das Produkt aus allen von
einander verschiedenen in \( m \) aufgehenden Primzahlen, und \( S_k \) die Summe der \( k \)ten
Potenzen aller primitiven Wurzeln der Gleichung \( z^m = 1 \), so ist \( S_k = 0 \), so oft \( k \) nicht durch \( m' \) teilverhältnis ist;
ist aber \( k = m'K \), ferner \( Q \) der größte gemeinsamliche Divisor von \( K \) und \( P = QR \), und \( r \)
die Anzahl der in \( R \) aufgehenden Primzahlen, so ist

\[ S_k = (-1)^r m' \phi(Q). \]

It is easy to see from this last relation that \( S_k = \Phi(k, n) \). To agree with the nota-
tion in our quoted article, we write

\[ C_n(k) = \sum_{(r,n)=1} \alpha^{kr}; \quad \alpha = e^{2\pi i/n}. \tag{2} \]

Von Sterneck\(^1\) employed (1) without any reference to (2). He defined (1) com-
pletely but without using the Möbius function explicitly. He found some proper-
ties of (1) and used it in investigating the number of different ways an integer \( s \) may
be expressed as the sum of \( t \) integers, if each of the summands involved is reduce
to its least residue modulo \( m > 1 \) and when no attention is paid to the order of the
summands.\(^2\)

The sums on the right of relation (2) are called "Ramanujan sums," since they
were considered in the year 1920 by Ramanujan. It is clear from the above that
the sums have been misnamed since that time, but, because considerable literature
has grown up concerning the sums, in which they are named after Ramanujan, it
seems too late to attempt to change the name.

In 1936 Hölder proved

\[ C_n(k) = \Phi(k, n) \tag{3} \]

explicitly and proved a number of related analytic results.

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We shall now consider giving names to (1) and (3). Of course, we could call (1) the Dedekind number, but he merely stated its connection with (2). The first to offer a proof appears to have been Hölder, and we think it appropriate to call (3) the "Dedekind-Hölder theorem." As to naming the \( \Phi(k, n) \) number itself, we think it appropriate to call it the "Dedekind–Von Sterneck number," as Von Sterneck discovered it in connection with something other than the Ramanujan sums and then obtained quite a number of properties of it, when you consider all the formulas he found involving it and partitions, modulo \( m \). We may point out that if the reader is disposed to call the relation (3) the "Dedekind theorem" or (1) the "Dedekind number," this may cause some confusion, as the latter's name has been connected with quite a number of concepts and results in algebra and number theory.

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When the order of the terms is taken into account in this problem, the authors found an expression also involving the \( \Phi \) numbers (op. cit., p. 833, relation [27]). This was generalized by Eckford Cohen, "An Extension of Ramanujan's Sum, II. Additive Properties," Duke Math. J., 22, 543–550, 1955, and "Some Totient Functions," ibid., 24, 515–522, 1956.

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ON THE NATURE OF THE SELECTIVE EFFECTS OF DEOXYRIBONUCLEIC ACID DIGESTS UPON PNEUMOCOCCI OF DIFFERENT VIRULENCE*

BY WILLIAM FIRSHEIN† AND WERNER BRAUN

INSTITUTE OF MICROBIOLOGY, RUTGERS, THE STATE UNIVERSITY, NEW BRUNSWICK, NEW JERSEY

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A previous communication\(^1\) described the ability of enzymatic digests of DNA (derived from either bacterial or mammalian sources) to promote bacterial population changes in vitro, resulting in the establishment of virulent, smooth (S) mutant cells in initially avirulent, non-smooth (R or M) populations. Such changes are ordinarily rare in vitro but are typical of the direction of changes known to occur in populations of pathogenic bacterial species in vivo, i.e., in susceptible hosts. It was shown that the addition of 150 \( \mu \)g DNA + 33\( \mu \)g DNAase/ml of growth medium (buffered beef-extract broth or brain-heart-infusion broth, respectively) promoted rapid R \( \rightarrow \) S population changes in either Brucella abortus or Diplococcus pneumoniae cultures. It has been established\(^2\), \(^3\) that in the case of Brucella these selective effects are due to the capability of S cells (but not of non-S cells) to convert a DNA breakdown product of relatively large molecular size\(^4\) into a toxic material for non-S cells. In contrast to such selection via selective inhibition, the mechanism responsible for R \( \rightarrow \) S population changes of pneumococci in the presence of DNA + DNAase (DD) now has been found to involve a selective stimulation of the growth of S cells, without any effects on R cells.

Since spontaneous mutation rates from R to S are exceedingly low in most R strains of pneumococci, inocula for in vitro cultures were prepared by mixing