ON IDENTICAL VANISHING OF HOLOMORPHIC FUNCTIONS IN
SEVERAL COMPLEX VARIABLES*

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We will make some comment on the following recent result of J. J. Kohn. In the
complex space \( C_n: (z_1, \ldots, z_n) \) if \( f_1, \ldots, f_n \) are holomorphic in the unit ball and
have continuous boundary values on the boundary, and if on the boundary the linear combination \( \bar{z}_1f_1 + \ldots + \bar{z}_nf_n \) is 0, then the functions \( f_1, \ldots, f_n \) are identically 0.

1. We denote by \( D \) any bounded circular domain in \( C_n \), by \( B \) its boundary, and
by \( B_0 \) an open neighborhood in \( B \). We do not assume that \( D \) contains the origin in
\( C_n \), but only that \( B_0 \) has a positive distance from it. For any integer \( m \), \( m \neq n \),
we take functions \( f_1, \ldots, f_m \) which are holomorphic in \( D \) and have continuous
boundary values on \( B_0 \). We also take \( m \) polynomials \( P_1, \ldots, P_m \) in \( n \) symbols
\( \xi_1, \ldots, \xi_n \), homogeneous and of a common degree and we denote the degree by \( r \).
Finally, we introduce the function

\[
\Phi(\xi; z) = \sum_{\mu=1}^{m} P_{\mu}(\xi)f_{\mu}(z)
\]

(1)
in the \( 2n \) variables \( \xi_1, \ldots, \xi_n; z_1, \ldots, z_n \). If for \( z \in D \) we make the specialization
\( \xi_v = \bar{z}_v, \nu = 1, \ldots, n \), then we obtain a function \( \Phi(z; z) \) which in \( D \) is analytic in
the \( 2n \) real variables \( x_1, \ldots, x_n; y_1, \ldots, y_n, z_{\nu} = x_{\nu} + iy_{\nu} \), and has continuous
boundary values on \( B_0 \).

THEOREM 1. (i) If the function \( \Phi(z; z) \) is 0 on \( B_0 \), then the function \( \Phi(\xi; z) \) is
identically 0 in \( (\xi; z) \).

(ii) Hence, if the polynomials \( P_1, \ldots, P_m \) are linearly independent over constants,
then \( f_1, \ldots, f_m \) are all identically 0.

Proof: In the case considered by Kohn, \( P_1 = \xi_1, \ldots, P_n = \xi_n \). These polynomials
are independent and his conclusion follows.

In our general case, let \( a = (a_1, \ldots, a_n) \) be a fixed point of \( D \), not the origin in
\( C_n \). If we put \( z_{\nu} = ta_{\nu}, \nu = 1, \ldots, n \), then we have

\[
\Phi(\bar{a}; ta) = \sum_{\mu=1}^{m} P_{\mu}(\bar{a})f_{\mu}(ta),
\]

(2)
and in the complex \( t \)-plane the latter sum is defined and holomorphic in a certain
annulus with center at \( t = 0 \). Because of our assumptions, there is a neighborhood
\( D_0 \) in \( D \) such that for each fixed \( a \in D \), the sum in equation (2) is 0 on an arc of the
\( t \)-annulus. Hence the sum is identically 0 in \( t \), and hence for \( z \in D_0, \Phi(z; z) \) is 0. By a
uniqueness theorem of E. Cartan, the function \( \Phi(\xi; z) \) is identically 0 as claimed.

2. In the case of the unit ball a statement can be made even if the polynomials
\( P_1, \ldots, P_n \) are arbitrary non-homogeneous, and the reader will readily verify that
this statement subsumes Theorem 1, for such a \( D \).

THEOREM 2. If \( D \) is the unit ball, and \( P_1, \ldots, P_m \) are any polynomials in \( \xi_1, \ldots, \xi_n \), then the vanishing of \( \Phi(z; z) \) on a neighborhood \( B_0 \) of the boundary of \( D \) implies
a representation

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\[ \Phi(\zeta; z) = (1 - \zeta_1z_1 - \ldots - \zeta_nz_n)\psi(\zeta; z) \]  
\[ \text{in which } \psi(\zeta; z) \text{ is a polynomial in } (\zeta_i) \text{ with coefficients which are holomorphic in } D. \]

\text{Proof.} If we put
\[ u = 1 - \bar{z}_1z - \ldots - \bar{z}_nz, \]
then \( D \) is defined by \( u > 0 \) and \( B \) by \( u = 0 \). If now we substitute
\[ \bar{z}_1 = \frac{1}{z}(1 - u - \bar{z}_2z_2 - \ldots - \bar{z}_nz_n) \]
in \( \Phi(\bar{z}; z) \) we obtain an expression \( A(u, \bar{z}_2, \ldots, \bar{z}_n; z_1, \ldots, z_n) \) which is a polynomial in \( u, \bar{z}_2, \ldots, \bar{z}_n \) with coefficients in \( z_1, \ldots, z_n \) which are holomorphic in a neighborhood \( D_0 \) in \( D \) bordering on an open part of \( B_0 \). For \( u = 0 \) the expression has value 0. Hence we can put
\[ A(u, \bar{z}_2, \ldots, \bar{z}_n; z) = uB(u, \bar{z}_2, \ldots, \bar{z}_n; z). \]
and if herein we replace \( u \) by the sum (4), we obtain a representation (3) to the following extent. The object (3) is a polynomial in \( (\zeta_i) \) with coefficients which are holomorphic in \( D_0 \), and the equality (3) is valid for \( z \in D_0 \) and \( \zeta_i = \bar{z}_i \). However, we can form the quotient
\[ \psi(\bar{z}; z) = \frac{\Phi(\bar{z}; z)}{1 - \bar{z}_1z_1 - \ldots - \bar{z}_nz_n} \]
in the entire unit ball \( D \), and it is analytic in the real variables \((x_i; y_i)\). Therefore, \( \psi(\zeta; z) \) has a holomorphic continuation from \( D_0 \) into \( D \), which completes the proof of the theorem.

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\[ ^1 \text{J. J. Kohn, } "\text{A Boundary Condition for the Vanishing of } n \text{ Holomorphic Functions in Complex } n\text{-Space}," \text{Proc. Am. Math. Soc.,} 9, 175-177, 1958. \]

\section*{LINEAR AND ALGEBRAICDEPENDENCEOFFUNCTIONSONCOMPACTCOMPLEXSPACESWITHSINGULARITIES*}

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We will axiomatize the following situation. Take a (non-compact) complex space \( W^m \), entirely regular, and form the quotient space \( V = W^m/R \) relative to some equivalence relation \( R \), say relative to the action of a group of homeomorphisms. It is the space \( V \) we are interested in. We assume that it is compact, but about its "regularity" (or what may be left of it) we only assume as follows. For some complex dimension \( n \), \( n \leq m \), there is in \( W^m \) a finite number of separate complex \( n \)-cells, each holomorphically immersed, such that the union of their pro-