LINE-ELEMENT FIELDS ON THE TORUS *

BY BRUCE L. REINHART

UNIVERSITY OF MICHIGAN

Communicated by Solomon Lefschetz, November 10, 1958

1. A line-element field on the two-dimensional torus $T^2$ is a cross-section in the unit tangent bundle (or the projective space bundle if we consider unoriented line elements). Since either of these bundles is the product bundle $T^2 \times S^1$, we may consider such a field as a mapping $F : T^2 \to S^1$, which we shall assume to be of class $C^1$. We shall parametrize $S^1$ as follows: Consider $T^2$ as the quotient of the plane $\mathbb{R}^2$ by the points with integer coordinates. At each point, let $\theta(v)$ be the angle between the vector $v$ and the positive $x$ axis. Then $\theta/2\pi$ will be taken as parameter on the fiber of the tangent sphere bundle and $\theta/\pi$ on the fiber of the tangent projective space bundle.

Let $F_* : H_1(T^2) \to H_1(S^1)$ be the induced homology mapping. Then for any closed curve $C$ on $T^2$ lying in the homology class $C^*$, $F_*(C^*) = \mu \xi$, where $\xi$ is the generator of $H_1(S^1)$ and $\mu$ is the winding number of $C$ with respect to the field $F$. The $C^1$ homotopy classes of mappings $F$ are in one-one correspondence with the induced maps $F_*$. In what follows, we shall be interested in the relation between the integral curves of $F$, considered as defining a differential equation on $T^2$, and the map $F_*$. 

**Lemma 1.** If $C$ is a closed integral curve, the homology class $C^*$ lies in the kernel of $F_*$ and, in fact, generates it if $F_*$ is not identically zero.

2. All the closed integral curves of $F$ belong to the same homology class, say $C^*$. If $\Gamma^*$ is any other homology class containing a simple closed curve, and $\Gamma \in \Gamma^*$ is a simple closed curve, it follows from elementary properties of the intersection ring of the torus and the Poincaré-Bendixson theorem that all integral curves of $F$ meet $\Gamma$.

**Lemma 2.** There exists a $C^1$ simple closed curve $\Gamma \in \Gamma^*$ which has everywhere a non-zero tangent vector and is tangent to $F$ at only a finite number of points.

Henceforth, let us denote by $\Gamma$ a curve in $\Gamma^*$ having the fewest possible points of tangency. Let $\mu(\Gamma)$ be the winding number of $\Gamma$ with respect to $F$.

**Theorem.** $\mu(\Gamma) = 0$ if and only if $F_* \equiv 0$. In particular, if there is no closed integral curve, these conditions hold.
Proof: If there exists a closed integral curve, then both $\Gamma^*$ and $C^*$ are in the kernel of $F_*$, so $F_*=0$. If there is no closed integral curve, let $\Gamma'$ be a cycle without contact.\textsuperscript{3} Any integral curve of $F$ meets $\Gamma'$ infinitely often; let $P$ be a point of accumulation of these intersections. We may then construct a $C^1$ closed curve $C''$ which differs from the integral curve only in an arbitrarily small neighborhood of $P$. $C'$ and $\Gamma'$ always cross in the same direction; hence their intersection number is non-zero, so that they neither bound nor are homologous. The winding number of $\Gamma'$ is zero because it is without contact, while that of $C''$ is zero because it differs from an integral curve by a small amount. Hence $F_*=0$ as before.

**Corollary 1.** If there exists on $T^2$ any curve whose winding number with respect to $F$ is non-zero, then there exists a closed integral curve of $F$.

3. Making use of the cycle $\Gamma$, we may generalize the rotation number $\lambda(\Gamma)$, which is classically defined for the case that there exists a cycle without contact.\textsuperscript{3, 4} Let $\Gamma$ and $\Gamma_1$ form a basis for $H_1(T^2)$, and let $F_*^{\Gamma}(\Gamma) = \mathfrak{i}\xi$, $F_*^{\Gamma}(\Gamma_1) = \mathfrak{j}\xi$.

**Proposition.** $\mu(\Gamma) = i$ and $\lambda(\Gamma) = -\mathfrak{j}/i$, the latter holding only in case $\mu(\Gamma) \neq 0$.

By considering the number of points of tangency on the cycle $\Gamma$, we may classify qualitatively the various kinds of integral curve families of non-oriented line-element fields.\textsuperscript{4} From this classification, the following corollaries are immediate:

**Corollary 2.** The non-oriented line element field $F$ is orientable if and only if $i$ and $j$ are both even.

**Corollary 3.** The number of closed integral curves is at least the greatest common divisor of $i$ and $j$.

\textsuperscript{4} This research was supported by the Office of Naval Research.


---

**ON A PROBLEM OF MAZURKIEWICZ CONCERNING THE BOUNDARY OF A COVERING SURFACE**

By Peter Seibert

A Member of Rias, 7212 Bellona Ave., Baltimore, Md.

Communicated by Solomon Lefschetz, November 13, 1968

1. In his note\textsuperscript{1} Mazurkiewicz showed that, by introducing a certain metric on a Riemann covering surface and completing the obtained metric space, a concept of "boundary elements" can be defined which corresponds precisely to the usual notion of transcendental singularities of meromorphic functions defined on the surface. This boundary, considered by itself, is a complete separable metric space. Moreover, Mazurkiewicz proved in the above-mentioned note that the dimension (in the sense of Menger-Urysohn) of the boundary never exceeds the value 2 and raised the question whether covering surfaces with two-dimensional boundaries exist. To our knowledge an answer to this question has, so far, not been given.