We postpone to another occasion the proof of

**Theorem 7.** Let the function $f(x)$ be of class $C^r$ on a closed bounded interval $E$, and let $p_n(x)$ ($\neq f(x)$ on $E$) be a polynomial of degree $n$ which for some $p$, $0 < p < 1$, minimizes (3). Let $f(x) - p_n(x)$ have zeros of respective multiplicities $\kappa_j$, $1 \leq j \leq r$, and let $\kappa_j^*$ be the smallest integer $\geq [(1 - p) \kappa_j]$ and which for zeros interior to $E$ is also of the same parity as $\kappa_j$. Then we have

$$\Sigma \kappa_j^* > n.$$  \hspace{1cm} (20)

As an illustration here, suppose all $\kappa_j$ are unity; at each end point of $E$ we have $\kappa_j^* = 0$, so (20) is precisely the condition that the number of strong sign-changes of $f - p_n$ be greater than $n$.

The conditions developed in Theorems 3, 6, and 7 are necessary but in the judgment of the writers (which is still to be confirmed) are close to sufficient.

There are numerous differences between approximation (i) on a closed bounded interval $E$ and (ii) on a real finite set $E_m$. In (ii) every juxtapolynomial is extremal with $p = 1$ and suitable weight function, but not in (i), as is shown by the example $f(x) = x^2$, $n = 0$, $E$: $0 \leq x \leq 1$; compare (19). In (i) the conditions derived become progressively stronger for juxtapolynomials and extremal polynomials with increasing $p$, namely $0 < p < 1$, $p = 1$, $p > 1$; in (ii) (for suitable weight function) the conditions are weakest for $p = 1$, stronger for $0 < p < 1$, still stronger for $p > 1$. In (ii) the conditions are independent of $p$, $0 < p < 1$, but not in (i). In (i) the extremal polynomial for $p = 1$ is unique, but not necessarily in (ii).

The present writers plan to continue this study, by considering further both necessary and sufficient conditions for best approximation to functions not necessarily continuous, using weight functions and more general real point sets.

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4 Jackson, D., Ibid., 320–326 (1921).

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**NOTE ON INVARIANCE OF DEGREE OF POLYNOMIAL AND TRIGONOMETRIC APPROXIMATION UNDER CHANGE OF INDEPENDENT VARIABLE**

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The object of this note is to show that various commonly used measures of degree of approximation are invariant under one-to-one analytic transformation of the independent variable, and this is true for both approximation by polynomials in the complex variable and trigonometric approximation in the real variable.
The methods used here have already been used several times in the literature, especially by J. H. Curtiss, Walsh and Sewell, and Walsh, but the present treatment is more unified and more complete, and emphasizes for the first time invariance of degrees of approximation. We call attention especially to Theorem 3 below, concerning trigonometric approximation.

The writer has recently established the essence of

**Theorem 1.** Let \( \epsilon_1, \epsilon_2, \ldots \) be a sequence of positive numbers approaching zero, where we suppose \( \epsilon_{(n/\lambda)} = O(\epsilon_n) \) for every positive integral \( \lambda \) (here \([m]\) denotes the largest integer not greater than \(m\)), and where for every \( r \), \( 0 < r < 1 \), we have \( r^* = O(\epsilon_n) \).

Let \( E \) be a region containing the origin in its interior, and let \( D \) be a region containing \( E \). Let the function \( f(z) \) be defined on \( E \), let the functions \( f_n(z) \) be analytic in \( D \), and let \((n = 1, 2, \ldots)\)

\[
|f(z) - f_n(z)| \leq A_1 \epsilon_n, \quad z \text{ on } E, \tag{1}
\]

\[
|f_n(z)| \leq A_2 R^n, \quad z \text{ in } D, \tag{2}
\]

be satisfied. Then there exist polynomials \( p_n(z, 1/z) \) in \( z \) and \( 1/z \) of respective degrees \( n \) such that

\[
|f(z) - p_n(z, 1/z)| \leq A_3 \epsilon_n, \quad z \text{ on } E. \tag{3}
\]

Here and below, the letter \( A \) with subscript denotes a positive constant independent of \( n \) and \( z \), a constant which may change from one usage to another.

The functions \( f_n(z) \) are analytic and satisfy (2) in some closed annular subregion \( D_1 \) of \( D \) bounded by two rectifiable Jordan curves \( J_1 \) to which \( E \) is interior and \( J_2 \) interior to \( E \), with \( O \) interior to \( J_2 \). For \( z \) interior to \( D_1 \) the function \( f_n(z) \) is represented as the sum of two integrals taken in the positive sense with respect to \( D_1 \):

\[
f_n(z) = f_n(z) + f_n(z), \quad z \text{ in } D_1, \tag{4}
\]

\[
f_n(z) = \frac{1}{2\pi i} \int_{J_1} f(t) dt, \quad f_n(z) = \frac{1}{2\pi i} \int_{J_1} f(t) dt \tag{5}
\]

where \( f_n(z) \) is analytic throughout the interior of \( J_1 \) and \( f_n(z) \) is analytic throughout the exterior of \( J_2 \), even at infinity. Let \( g(z) \) denote Green’s function with pole at infinity for the exterior of \( E \), and let the locus \( g(z) = \log \sigma \) (\( \sigma > 1 \)) be generically denoted by \( C_* \); let \( g_0(z) \) denote Green’s function with pole at 0 for the interior of \( E \), and let the locus \( g_0(z) = \log \sigma \) (\( \sigma > 1 \)) be generically denoted by \( C_*^0 \). From (2) and (5) follow the inequalities

\[
|f_n(z)| \leq A_4 R^n, \quad |f_n(z)| \leq A_4 R^n, \tag{6}
\]

respectively on any closed set interior to \( J_1 \) and on any closed set exterior to \( J_2 \); we choose these sets to be closed Jordan regions containing \( E \) bounded respectively by suitably chosen \( C_* \) and \( C_*^0 \).

There exists ([§4.5 of reference 1]) for every \( n \) a sequence of polynomials \( P_n(z) \) in \( z \) of respective degrees \( k = 1, 2, \ldots \), defined by interpolation to \( f_n(z) \) in points of some \( C_* \) near \( E \), and a sequence of polynomials \( Q_n(1/z) \) in \( 1/z \) of respective degrees
k, defined by interpolation to \( f_{n2}(z) \) in points of some \( C_\rho^6 \) near \( E \), such that we have by (6)

\[
|f_{n2}(z) - P_{nk}(z)| \leq A_5 R^n/\rho_1^k, \quad z \text{ on } E, \quad \rho_1 > 1,
\]

\[
|f_{n2}(z) - Q_{nk}(1/z)| \leq A_5 R^n/\rho_1^k, \quad z \text{ on } E;
\]

where \( A_5 \) is independent of \( k, n, \) and \( z \).

From (4) and (7) we may now write

\[
|f_n(z) - p_{n,\lambda n}(z, 1/z)| \leq A_6 (R/\rho_1^\lambda)^n, \quad z \text{ on } E,
\]

with \( p_{nk}(z, 1/z) \equiv P_{nk}(z) + Q_{nk}(1/z) \) and with the positive integer \( \lambda \) so chosen that \( \rho_1^\lambda > R \). By (1) and the properties of the \( \epsilon_n \) we now have

\[
|f(z) - p_{n,\lambda n}(z, 1/z)| \leq A_7 \epsilon_n. \tag{8}
\]

The polynomials \( p_{n,\lambda n}(z, 1/z) \) are defined only for the degrees \( j = \lambda, 2\lambda, 3\lambda, \ldots \), but if we set

\[
p_j(z, 1/z) \equiv p_{n,\lambda n}(z, 1/z), \lambda n \leq j < \lambda(n + 1),
\]

\[
p_j(z, 1/z) \equiv 0, \quad 1 \leq j < \lambda,
\]

the polynomials \( p_j(z, 1/z) \) are defined for all \( j \geq 1 \), and (3) follows from (8).

A modification of Theorem 1 is of independent interest:

**Corollary.** If in Theorem 1 the functions \( f_n(z) \) are analytic also throughout the interior of \( E \) (that is, if \( D \) contains \( E \) and its interior), then the polynomials \( p_n(z, 1/z) \) in (3) may be chosen as polynomials in \( z \).

In the proof we set \( f_{n2}(z) \equiv 0, Q_{nk}(1/z) \equiv 0 \).

It will be noticed that the hypothesis of Theorem 1 is invariant under suitable one-to-one conformal transformation of the region \( D \). Moreover, if the \( f_n(z) \) satisfying (1) are given as polynomials of degree \( n \) in \( z \) and \( 1/z \), then [Lemma, p. 259 of reference 1] (2) is a consequence of (1). There follows

**Theorem 2.** Let the \( \epsilon_n \), \( E, D, \) and \( f(z) \) satisfy the conditions of Theorem 1. If \( f(z) \) can be approximated by polynomials in \( z \) and \( 1/z \) of respective degrees \( n \) in \( E \) with degree of approximation \( O(\epsilon_n) \), if \( D \) is mapped one-to-one and conformally onto a region \( D_1 \) of the \( w \)-plane with \( E \) transformed into \( E_1 \), containing \( w = 0 \) in its interior and if \( f(z) \) is transformed into \( f_1(w) \), then \( f_1(w) \) can be approximated by polynomials in \( w \) and \( 1/w \) of respective degrees \( n \) on \( E_1 \) with degree of approximation \( O(\epsilon_n) \). Thus the class of functions \( f(z) \) which can be approximated by polynomials in \( w \) and \( 1/w \) on \( E_1 \) with degree of approximation \( O(\epsilon_n) \) is identical with the class of the transforms of functions \( f(z) \) which can be approximated by polynomials in \( z \) and \( 1/z \) on \( E \) with degree of approximation \( O(\epsilon_n) \).

In particular, if \( D \) contains both \( E \) and the interior of \( E \), only polynomials in \( z \) and in \( w \) respectively may be involved.

If \( E \) and \( E_1 \) are arbitrary analytic Jordan curves, there exist regions \( D \) and \( D_1 \), and an infinity of transformations of the kind required, even if \( D \) contains both \( E \) and its interior. Either \( E \) or \( E_1 \) may be chosen as the unit circle:

**Corollary 1.** Let \( E \) be an analytic Jordan curve of the \( z \)-plane containing \( z = 0 \) in its interior, let the \( \epsilon_n \) satisfy the conditions of Theorem 1, and let the image of \( E \) under a one-to-one conformal map be \( E_1 \): \( |w| = 1 \). The class of functions \( f(z) \) that can be
approximated on \( E \) by polynomials in \( z \) and \( 1/z \) of respective degrees \( n \) with degree of approximation \( O(\varepsilon_n) \) is identical with the class of the transforms of functions \( f_1(w) \) that can be approximated on \( E_1 \) by polynomials in \( w \) and \( 1/w \) of respective degrees \( n \) with degree of approximation \( O(\varepsilon_n) \).

In particular, if the map transforms the interior of \( E \) onto the interior of \( E_1 \) one-to-one and conformally, only polynomials in \( z \) and in \( w \) respectively may be involved.

Choice of \( E \) in Corollary 1 as \( |z| = 1 \) gives

Corollary 2. Under the conditions of the first or second part of Corollary 1, the class of functions \( f(z) \) involved is invariant under a one-to-one conformal transformation which maps \( E_1 \) onto itself.

It is readily shown [compare Lemma 2 of reference 4] that a one-to-one analytic map with nonvanishing derivative of an analytic Jordan curve onto another is one-to-one and analytic in suitable regions containing those curves in their respective interiors.

Corollaries 1 and 2 are of especial significance in that they refer to the unit circle \( E_1: |w| = 1 \), on which any polynomial in \( w \) and \( 1/w \) of degree \( n \) is also a trigonometric polynomial in \( \theta = \arctan w \) of order \( n \), and conversely, by virtue of the Euler relations \( w^k = e^{i\theta} = \cos k\theta + i \sin k\theta \). Moreover approximation on \( E_1 \) to a real function by polynomials in \( w \) and \( 1/w \) of degree \( n \) is equivalent to approximation on \( E_1 \) by real trigonometric polynomials of order \( n \), with the same degree of approximation. Thus we have

**Theorem 3.** If \( \varepsilon_n \) satisfies the conditions of Theorem 1, the class of real functions of period \( 2\pi \) that can be approximated by trigonometric polynomials of respective orders \( n \) with degree of approximation \( O(\varepsilon_n) \) is invariant under one-to-one analytic map \( x' = \phi(x) \) of the axis of reals onto itself with nonvanishing derivative and with \( \phi(x + 2\pi) = \phi(x) + 2\pi \).

Further, if this map is of power series type when interpreted on the unit circumference, the class of functions of period \( 2\pi \) that can be approximated by trigonometric polynomials of respective orders \( n \) of power series type with degree of approximation \( O(\varepsilon_n) \) is invariant.

If in Theorem 3 we have \( \varepsilon_n = 1/n^\alpha + a \), where \( p \) is a nonnegative integer and \( 0 < \alpha < 1 \), the class of functions \( f(x) \) involved is the class for which \( f^{(p)}(x) \) exists and satisfies a Lipschitz condition in \( x \) of order \( \alpha \), as is known by the Bernstein-Jackson-Montel-de la Vallée Poussin theory; for this class the invariance is easy to prove. But with such a choice as \( \varepsilon_n = 1/n \log n \) (\( n \geq 2 \)), the invariance is by no means obvious. Zygmund has introduced special classes of functions characterized by \( \varepsilon_n = 1/n^\alpha + 1 \) in Theorem 3, namely the class of functions possessing continuous derivatives of the \( k \)th order satisfying a uniform condition

\[
|F(x + h) + F(x - h) - 2F(x)| \leq A|h|;
\]

the invariance also of these classes follows from Theorem 3. The analogous classes on an arbitrary analytic Jordan curve were introduced and studied by Walsh and Elliott.

**Theorem 1** refers to the Tchebycheff measure of approximation both in hypothesis (1) and conclusion (3). However, the conclusion persists if (1) is replaced by the inequality

\[
\int_E |f(x) - f_n(x)|^p |dx| \leq A_1 \varepsilon_n^p, \quad p > 0,
\]
and (3) replaced by the corresponding inequality

$$\int_E |f(z) - p_n(z, 1/z)|^p |dz| \leq A_2 \epsilon_n^p,$$

(10)

where of course we assume $E$ rectifiable. In the proof, no essential change needs to be made, although we use the standard inequalities

$$|x_0 + x_1|^p \leq 2^{p-1}(|x_0|^p + |x_1|^p), \quad p > 1,$$

$$|x_0 + x_1|^p \leq |x_0|^p + |x_1|^p, \quad 0 < p \leq 1;$$

the analog of the Corollary persists.

Likewise Theorem 2, Corollary 1, Corollary 2, and Theorem 3 have valid analogs involving integral measures of approximation as in (9) and (10). In the proof we need to derive (2) as a consequence of such an inequality as (10); here the above standard inequalities yield

$$\int_E |p_n(z, 1/z)|^p |dz| \leq A_0,$$

(11)

whence [Lemma of reference 2] we have

$$|p_n(z, 1/z)| \leq A_1\rho^\alpha, \quad \rho > 1,$$

(12)

for $z$ in the annular region bounded by the curves $C_\rho$ and $C_\rho^0$ used in the proof of Theorem 1.

Hardy and Littlewood have introduced classes of functions whose $k$th derivatives satisfy integrated $p$th power ($p \geq 1$) Lipschitz conditions of order $\alpha$, $0 < \alpha < 1$; if $E$ is analytic these classes are characterized by (10) with $\epsilon_a = 1/n^k + \alpha$. Proofs in the trigonometric case were provided by Quade, in the case of analytic functions (where the integrated Lipschitz conditions are taken with respect to arc length) by Walsh and Russell.4 The proof of (10) from (9) establishes anew4 the invariance of these classes under suitable conformal transformation. There are corresponding results for trigonometric and polynomial approximation to functions whose $k$th derivatives satisfy an integrated $p$th power Zygmund condition, in the case of polynomial approximation with respect to arc length; the class of functions is likewise invariant.

If the Corollary to Theorem 1 is modified so that (1) and (3) are replaced not by inequalities involving line integrals as in (9) and (10) but surface integrals:

$$\int_{E^*} |f(z) - f_n(z)|^p dS \leq A_1 \epsilon_n^p, \quad p > 0,$$

(13)

$$\int_{E^*} |f(z) - p_n(z)|^p dS \leq A_3 \epsilon_n^p,$$

(14)

taken over the interior $E^*$ of $E$, the conclusion persists. Likewise the analogs of the latter part of Corollary 1 to Theorem 2 and of the latter part of Theorem 3 are valid. In the modified proof we derive

$$\int_{E^*} |p_n(z)|^p dS \leq A_4,$$

(15)
as a consequence of (14), whence [§5.3, Lemma 2; §2.1, Theorem 2; §4.60 Lemma, all of reference 1] we have

$$|p_n(z)| \leq A_1 \sigma_1^n, \quad 1 < \rho < \sigma_1,$$

(16)

throughout the interior of the curve $C_\rho$ used in the proof of Theorem 1.

In the important case $p = 2, E: |z| = 1$, the sum of the first $n + 1$ terms of the Maclaurin development of $f(z)$, with the hypothesis of the second part of Corollary 1 to Theorem 2, is the polynomial $p_n(z)$ of degree $n$ of best approximation to $f(z)$ on both $E$ and $E^*$. There is then a close relation6 between the two measures of best approximation, say the first members $M_n^2$ and $M_n^{*2}$ of (10) and (14), namely

$$M_n^{*2} \leq M_n^2 (n + 2).$$

(17)

If $\mu_n^2$ and $\mu_n^{*2}$ denote the corresponding measures of approximation after conformal map of $E^*$ onto the interior of an analytic Jordan curve, we have (if $M_n$ and $M_n^*$ satisfy the conditions of Theorem 1 on $\epsilon_n$)

$$\mu_n^{*2} \leq A_1 M_n^{*2} \leq A_1 M_n^2 (n + 2) \leq A_1 \mu_n^2 (n + 2).$$

(18)

Thus (17) is invariant under conformal transformation, subject to an additional constant factor on the right.

The derivation of (17) and (18) also applies to approximation on $E: |z| = 1$ and $E^*$ by harmonic polynomials, which on $E$ is equivalent to approximation by trigonometric polynomials.

Throughout this paper, positive continuous weight functions can be inserted in such inequalities as (1), (3), (9), (10), (13), (14), (17), (18) without altering the truth of our conclusions.

Dedicated to Marston Morse as one of a series of articles so dedicated by a number of associates cooperating to honor his mathematical achievements.

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**BIOASSAY OF ORGANIC MICRONUTRIENTS IN THE SEA**

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**SCIRPPS INSTITUTION OF OCEANOGRAPHY**

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**Introduction.**—It has often been stated1 that oceanic production is limited by the quantity of phosphate in the marine environment. The best data available indicate that the concentration of phosphate in the water off the coast of California is about ten times as high as that found in the English Channel. Yet the total productivity