ORBIT SPACES OF ABELIAN $p$-GROUPS*

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1. Let $p$ be a prime and $S$ a cohomology $n$-sphere over $Z_p$ (the integers mod $p$). Let $\pi_a$ be an abelian group of order $p^a$ and type $(p, \ldots, p)$ acting effectively on $S$ to form a transformation group $(S, \pi_a)$. Let $S_f$ be the $\pi_a$-free part of $S$, that is, the union of orbits of cardinality $p^a$. For $\pi_b \subseteq \pi_a$, denote by $F(\pi_b)$ the fixed-point set of $\pi_b$. $F(\pi_b)$ is a cohomology sphere over $Z_p$. Let $n(\pi_b) = \dim Z_p F(\pi_b)$. Then $n(\pi_b) \leq n$ and if $p \neq 2$, $n - n(\pi_b)$ is even. Cyclic subgroups $\pi_1$ of $\pi_a$ which are not in $\pi_b$ act effectively on $F(\pi_b)$. If $U$ is $\pi_b$-invariant, denote by $P(U, \pi_b^i) = P(U, \pi_b, t)$ the Poincaré polynomial of the cohomology of the orbit space $U/\pi_b$ over $Z_p$. It is known$^1$ that for $a = 1$, $P(S_f, \pi_1) = P(n(\pi_1) + 1, n)$ (1.1) where $P(h, k) = t^h + t^{h+1} + \ldots + t^k$. We conjecture that if $a \geq 2$, then for any subgroup $\pi_{a-1}$ of $\pi_a$, $(1 - t)P(S_f, \pi_a) = \sum_{\pi_i \subseteq \pi_{a-1}} P(F(\pi_i), \pi_{a-1}) - tP(S_f, \pi_{a-1})$ (1.2) where,$^2$ in $P(A_f, \mu)$, $A_f$ means the $\mu$ free part of $A$. Since $\pi_a$ is abelian, $\pi_{a-1}$ acts on $(F(\pi_1))$. We shall sketch a proof for the case $a = 2$. It can be shown that for $a \geq 2$, (1.2) implies the relations $\sum_{\pi_{a-1} \subseteq \pi_a} (n(\pi_{a-1}) - n(\pi_a)) = n - n(\pi_a)$ (1.3) which have been established by Borel.$^3$

From now on take $a = 2$ and denote $\pi_2$ by $\pi$. The subgroups $\pi_{a-1} = \pi_1$ are cyclic, $p + 1$ in number; denote them by $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_{p+1}$. Let $n_i = n(\alpha_i)$, $n_0 = n(\pi)$. The formula to be proved is $(1 - t)P(S_f, \pi) = \sum_{i = 2}^{p+1} P(F(\alpha_i), \alpha) - tP(S_f, \alpha)$ which, as a consequence of (1.1) and the fact that the fixed-point set of $\alpha_i$ ($i \geq 2$) acting on $F(\alpha)$ is precisely $F(\pi)$, can be written $(1 - t)P(S_f, \pi) = \sum_{i = 2}^{p+1} P(n_i + 2, n_i + 1) - tP(n_i + 1, n)$ (1.4) $\sum_{i = 1}^{p+1} P(n_i + 2, n_i + 1) - P(n_0 + 2, n + 1)$ (1.4)$'$

The following explicit formula can be obtained readily from (1.4):

$$P(S_f, \pi) = t^n + 2 \sum_{i = 2}^{p+1} P(0, q_1 + \ldots + q_{i-1} - 1)P(0, q_i - 1),$$

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where \( q_i = n_i - n_0 \). The relations (1.3) for \( a = 2 \) come from putting \( t = 1 \) in (1.4).

2. An exact sequence. Let \( \beta \) be a cyclic group of order \( p \) and \( g \) a generator. In the group ring \( Z_\beta(\beta) \) let \( \sigma = 1 + g + \ldots + g, \tau = 1 - g. \) We have the relations \( \sigma \tau = \tau \sigma = \sigma \tau = 0. \) Now let \( Y \) be a free \( Z_\beta(\beta) \)-module which is a cochain complex and let \( Y^\beta \) denote the set of \( \beta \)-invariant elements. Evidently \( Y^\beta = \sigma Y. \)

Let \( \rho \) be either one of \( \sigma, \tau \) and \( \bar{\rho} \) the other. From now on, every statement involving \( \rho, \bar{\rho} \) will hold equally when \( \rho, \bar{\rho} \) are interchanged.

There exists an exact sequence

\[
0 \rightarrow \rho Y \rightarrow Y \rightarrow \bar{\rho} Y \rightarrow 0
\]

giving rise to the cohomology sequence

\[
\ldots \rightarrow H^k_{\rho^{-1}} \rightarrow H^k_{\rho} \xrightarrow{i^*} H^k \xrightarrow{\delta^*} H^k_{\rho} \rightarrow H^k_{\rho +1} \rightarrow \ldots \tag{2.1}
\]

where \( H^k = H^k(Y), H^k_{\rho} = H^k(\rho Y), \ldots. \) Let \( b^k_{\rho} = \dim H^k_{\rho}, c^k_{\rho} = \dim(\ker \rho^{*k}), \delta^k_{\rho} = \dim(\im \rho^{*k}). \) Then

\[
b^k_{\rho+1} = b^k_{\rho} + c^k_{\rho+1} - \delta^k_{\rho},
\]

which, putting \( B_{\rho} = \sum b^k_{\rho} t^k \) etc. can be written

\[
tB_{\rho} = tB_{\bar{\rho}} + C_{\rho} - tE_{\rho}. \tag{2.2}
\]

Note that \( B_{\sigma} \) is the Poincaré polynomial (or series) for \( H(\sigma Y) = H(Y^\sigma) \).

3. If \( H^* = Z_p \), the composition of the maps

\[
H^* \xrightarrow{e^{*n}} H^*_p \xrightarrow{i^{*n}} H^*
\]

is zero. For let \( u \in H^* \) and let \( z \) be a cocycle in \( u \).

Then \( \rho z \) represents the image of \( u \) in \( H^*_p \), hence it also represents the image of \( u \) under the composition map. Since \( H^* = Z_p \), \( \beta \) acts trivially on \( H^* \), hence \( \rho z \) cobounds, so \( \rho z \) represents the zero of \( H^* \).

4. The cohomology of \( S_f/\alpha \). Let \( U_f, U_h \) denote the \( \pi \)-free and \( \alpha \)-free parts of \( U \) (see §1). From now on the index \( \iota \) will be restricted to the range \( 2, \ldots, p + 1. \)

Let \( F_i = F_h(\alpha_i), F = \cup F_i. \) One sees that the \( F_i \) are disjoint and so \( H(F/\alpha) = \Sigma H(F_i/\alpha), \) direct sum. Moreover,

\[
S_f = S_h - F, S_f/\alpha = S_h/\alpha - F/\alpha.
\]

Using (1.1), the cohomology sequence for the pair \( (S_h/\alpha, F/\alpha) \) yields

\[
H(S_f/\alpha) = \iota^* H + \Sigma \iota^* H; \quad \iota^* H = i^* H(S_h/\alpha); \quad \iota^* H = \delta^* H(F^1/\alpha) \tag{4.5}
\]

where \( i^* \) is an isomorphism and \( \delta^*, \) the connecting homomorphism, raises dimensions by 1 and is injective. The easiest way to verify this is to assume that \( n_i \leq n_1 \) which is permissible since the final result (1.4)' is independent of the order of the \( \alpha \)'s.

5. There is a natural effective action of \( \beta = \pi/\alpha \) on \( S_f/\alpha \) and it is readily verified that the fixed-point set of \( \beta \) is \( F/\alpha. \) We assert that each element \( w \) of \( \iota^* H \) contains a cocycle which \( \beta \)-invariant. It is sufficient to show that \( w \in pr^* H((S_f/\alpha)/\beta) \) where \( pr \) is the natural mapping \( S_f/\alpha \rightarrow (S_f/\alpha)/\beta \) and this last follows from the commutative
\[ H(F/\alpha) \xrightarrow{z^*} H(S_{\alpha} - F/\alpha) = H(S_{\alpha}/\alpha) \]

\[ H((F/\alpha)/\beta) = H(F/\alpha) \rightarrow H((S_{\alpha}/\alpha)/\beta - F/\alpha) = H(S_{\alpha}/\beta) \]

As a consequence of the preceding paragraph and the relation \( \rho \sigma = 0 \) (see §2) we have that

\[ \rho^*(\langle H \rangle) = 0, \quad i = 2, \ldots, p + 1, \quad (5.1) \]

where \( \rho^* = \Sigma \rho^* \).

6. We use \( S_f \) equally to denote the Alexander-Spanier cochains of \( S_f \) with compact carriers and values in \( Z_p \). Let \( W = C \otimes S_f \) where \( C \) is an acyclic \( \alpha \)-free cochain complex over \( Z_p(\alpha) \) on which \( \beta \) acts trivially.\(^4\) Let \( \alpha \) and \( \beta \) act diagonally on \( W \).

Now let \( z \) be a cocycle of \( C^n \). The endomorphism \( \xi \) of the complex \( W \) defined by \( \xi(\Sigma c_j \otimes s_j) = \Sigma z \cdot c_j \otimes s_j \) where \( z \cdot c_j \) is the cup product, permutes with \( \alpha \) and therefore defines an endomorphism of \( W^* \). One verifies that \( \xi \) permutes or antipermutes with the differential operator (depending on the degree of \( z \)), hence induces an endomorphism of \( H(W^*) \). Now \( \beta \) acts on \( W^* \) and is seen to permute with \( \xi \). Hence \( \rho^* \) (defined for \( Y = W^* \), §2) permutes with \( \xi \). It follows that

\[ \rho^* w = 0 \text{ implies } \rho^*(\xi w) = 0, \quad (w \in H(W^*)) \quad (6.1) \]

7. From now on take \( Y \) in §2 to be \( W^* \). From the known facts about tensor products\(^6\) we have

\[ H(W) = H(S_f); \quad H(Y) = H(S_f^\alpha) = H(S_f/\alpha); \]

\[ H(Y^\beta) = H(S_f^\beta) = H(S_f/\beta). \]

Since \( Y^\beta = \sigma Y \) we can compute \( H(S_f/\pi) \) as \( H(\sigma Y) \), and \( P(S_f, \pi) \) as \( B_*(\pi) \).

We first obtain \( H(Y) \). A simple spectral sequence argument\(^6\) yields an additive base for \( H(Y) \) of the form

\[ y^{n+1} = c^* \otimes u^{n+1} \quad s = 0, \ldots, n - n_1 - 1 \quad (7.1) \]

\[ i_y^{n+1} = c^* \otimes i u^{n+1} \quad s = 0, \ldots, n_1 - n_0 - 1, \quad i = 2, \ldots, p + 1, \quad (7.2) \]

where \( c^* \in H^*(C^n), \: u^{n+1} \in H^{n+1}(s_f), \: i u^{n+1} \in H^{n+1}(s_f) \). There exists\(^7\) in the algebra \( H(C^n) = H(\alpha) \) an element \( v = v(s) \) of degree \( n - n_1 - 1 - s \) such that \( v \cdot c^* \) is a nonzero element of degree \( n - n_1 \) - 1 (when \( p \neq 2 \), this depends on the fact that \( n - n_1 \) is even so that one of the integers \( s, n - n_1 - 1 \) is even). Let \( \pi(s) = 0 \) be a cocycle in \( v(s) \). We then have

\[ \xi v \cdot y^{n+1} = \lambda y^n \quad (0 \neq \lambda \in Z_p). \quad (7.3) \]

8. We next show that for \( Y = W^* \),

\[ H_p^\alpha = H_p^\beta = Z_p, \quad \text{Im} \: \rho^* = \text{Im} \: \bar{P}^* = Z_p. \quad (8.1) \]

For \( j \) sufficiently large, the groups \( H^j_p = H(Y^\beta) = H(S_f/\pi) \) are trivial since \( S_f/\pi \) is finite dimensional because \( S_f \) is. Hence from (2.1), \( H_p^j = H_p^\beta = 0 \) for \( j > n \), and \( \rho^* \), \( p^* \) are surjective. Suppose \( \rho^* = 0 \). Then in (2.1), \( i^* \) is surjective. Hence
the composition map in (3.1) is surjective. The composition map can therefore not be zero since (7.2) shows that $H^n(Y) = \mathbb{Z}_p$. Since we thus obtain a contradiction to the statement in §3, we conclude that $\rho^n_0 \neq 0$, $\bar{\rho}^n \neq 0$. It follows that $H^n \neq 0$, $H^{n+1}_\rho \neq 0$ and (8.1) now follows from $H^n = Z, H^{n+1} = H^{n+1}_\rho = 0$.

9. Determination of $C$ and $E$. Referring to the base (7.1) (7.2) we assert that $\rho^*(i^{y_i} + n + 1) \neq 0$; $\rho^*(i^{y_i} + n + 2) = 0$. (9.1)

Suppose $\rho^*(i^{y_i} + n + 1) = 0$. Then by (6.1) $\rho^*(\xi_i^{(i)}y^{i+1} + n + 1) = 0$, i.e., $\rho^*y^n = 0$ which, since $y^n$ is a generator of $H^n(Y)$ contradicts (8.1). This proves the first part of (9.1). As for the second, $\rho^*(i^{y_i} + n + 2) = \rho^*(c^* \otimes i^{u_{m+2}}) = c^* \otimes \rho^*(i^{u_{m+2}}) = 0$ by (5.1).

Consequence of (7.1) (7.2) (9.1) (see §2 for notation):

\[ c_p^k = \sum_i c^k_i, i_c^k = 1, k = n_0 + 2, \ldots, n_i + 1 \]
\[ e^k_p = 1, k = n_1 + 1, \ldots, n \quad (\rho = \sigma, \tau). \]

All other $c$'s and $e$'s are zero. It follows that

\[ C_\rho = C_{\bar{\rho}} = \sum_i P(n_0 + 2, \ldots, n_i + 1) \]
\[ E_\rho = E_{\bar{\rho}} = P(n_1 + 1, \ldots, n) \]

Formula (1.4) now follows from (2.2) and the fact that

\[ P(Sf, \pi) = B_\pi \text{ as remarked in §7.} \]

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1 We follow the definitions in Borel, “Novelle démonstration d’un théorème de P. A. Smith,” Commentarii Math. Helv., 29 (1955). We assume implicitly that the cohomology relations $H(B \mod A) = H(B - A)$ hold for certain pairs ($A, B$) constructed from $S$. It would for example be sufficient to assume that $S$ is metrizable, but certain weaker conditions can be used. Cohomology is based on Alexander-Spanier cochains with compact supports.

2 We write $F(f)$ for $(F(f))_f$. In $P(A_f, \pi)$, $A_f$ is the $\pi$-free part of $A$.


4 Tensor products are over $\mathbb{Z}_p$.


6 Let $E_\omega$ be the spectral sequence for the covering $S_f \to S_f / \alpha$. Then $E_\omega$ is associated with $H(Y^\omega)$. The cocycles listed form a base for a subgroup of $E_\omega$ which consists of permanent cocycles and projects isomorphically onto $E_\omega$.

7 See for example the reference in footnote 1.