(1) \( H_n(X; \mathbb{Z}) = \mathbb{Z} \)
(2) \( H_{n-1}(X; \mathbb{Z}) \) is free

(3) if \( \nu \) is a generator of \( H_1(X; \mathbb{Z}) \) then \( \cap \nu: H^0(X; \mathbb{Z}) \to H^1(X; \mathbb{Z}) \) is an isomorphism for all \( q \).

In this sense, finite dimensional \( H \)-spaces are homologically like manifolds. There is apparently no known example of a finite dimensional \( H \)-space which is not the homotopy type of a manifold.

* The author is a National Science Foundation Postdoctoral Fellow.


4. This construction also occurs in M. Nakaoka: “Cohomology theory of a complex with a transformation of prime period and its applications,” Journal of Inst. of Polytechnics, Osaka City University, vol 7, nos. 1 and 2 (1956).


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**ON THE USE OF ABSTRACT ALGEBRA AS A METHOD FOR OBTAINING RESULTS INVOLVING RATIONAL INTEGERS ONLY AND THE REVERSE OF THIS PROCEDURE**

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**Communicated February 16, 1960**

In the subject of number theory, of course, many parts of mathematics have been employed in deriving theorems which involve in their statements rational integers only. We may divide, rather roughly, these contributions under the general headings of modern algebra, geometry, and analysis. Conversely, different number theoretic problems and patterns have led mathematicians to new results in many parts of pure and applied mathematics.

In the present paper we shall be concerned mainly with the applications of modern algebra to number theory, particularly the number theory of rational integers as well as the results obtained in modern algebra which seem to have been suggested by arithmetic patterns. (An eminent example is the formulation of the concept of rings, which seems likely to have originated from the study by leading mathematicians of the properties of the set of algebraic integers in an algebraic field.) Under the heading of “modern algebra” we have included the classical theory of algebraic numbers, but we shall pay more attention to those parts of present-day algebra which are not regarded as belonging to said theory, such as semigroup and group theory, finite rings and fields, matrices and other noncommutative algebras, etc. As to our title, more than half the papers the writer has written on number theory, including algebraic numbers, contain in them results of this character. Obviously, then, such subjects are very close to him.
Up to approximately ten years ago, it seems that if we exclude elementary number theory, then by far the greater portion of articles published under the heading of number theory belong to analytic number theory. However, it seems that within the last ten years the proportion of articles based on advanced algebraic methods has been gradually increasing. We shall not discuss results here, however, the proofs of which use both analysis and modern algebra, or analysis alone.

The present paper constitutes only a beginning in our investigation of the relations between abstract algebra and number theory, and we have not as yet attempted to locate all the literature bearing on this topic. We hope eventually to publish more papers along this line.

Our principal aim in pursuing the present investigation is to isolate, if possible, certain tools in abstract algebra or the theory of algebraic numbers which seem to have been the most successful in deriving results concerning the rational integers alone. Comparison of the articles where such methods have been used for this purpose should then, perhaps, lead to important new ideas and results.

In our selection of results described in what follows, we probably give a bit more space to the statements of what seem to be curious or unusual relations rather than to the discussion of methods and theorems which are rather well known. In line with this, for example, we do not give much attention to the parts of number theory in which the development depends mainly on the use of linear transformation. We shall, however, try to give something of the historical background of all the subjects treated.

It will be convenient to begin our more detailed discussion by setting up certain distinctions. We shall define elementary algebra and elementary number theory as these two subjects were known prior to about 1770, and so it seems it would be proper to start with Lagrange⁶ in our discussion of the development of algebra and its application to number theory, as he appears to have been the first to employ linear substitutions in connection with algebraic or arithmetic forms. As to the latter, he investigated binary quadratic forms, using this tool in a systematic manner, and obtained important results.

Gauss made fundamental contributions to number theory by the use of algebraic methods other than elementary. This he did in particular in Chapter V in his Disquisitiones Arithmeticae, 1801, by developing the ideas of linear substitutions in connection with binary and ternary quadratic forms. Further, in Chapter VII, using a theory of the algebraic solution of the equation \( x^n = 1 \), given by Lagrange⁷ (the theory now usually known by the name cyclotomy (Kreisteilung)), he obtained the various arithmetical and geometrical results particularly pertaining to the construction of a regular polygon with 17 sides. He also obtained curious and remarkable relations between trinomial congruences, binomial coefficients, and certain binary quadratic forms. One of these⁸ is as follows: If \( p \) is a prime in the form \( 4n + 1 \) and \( p = x^2 + y^2 \),

\[
x \equiv \frac{1}{2} \frac{2n!}{(n!)^2} \mod p; \quad x \equiv 1, \mod 4,
\]

and a somewhat similar explicit form for the residue of \( y \), modulo \( p \). This result is extended by others, including Stickelberger⁹ and Hasse-Davenport.¹⁰
In the famous memoirs of Gauss\textsuperscript{11} on biquadratic reciprocity, he made the first advances toward what we now call the classical theory of algebraic numbers by defining an integer in the field generated by $i$ with $i^2 = -1$ as an expression of the form $a + bi$ with $a$ and $b$ rational integers. He then stated, without proof, the law of biquadratic reciprocity, which involves the use of these complex numbers. The relation of this theorem to the theory of rational integers may be described as follows.

In the theory of rational integers, for example, we are enabled, using an algorithm based on the law of quadratic reciprocity, to find the integers $x$, if any exist, such that $x^2 \equiv a \pmod{p}$ with $p$ prime. Similarly, by the use of an algorithm based on biquadratic reciprocity, we can determine the integers such that $y^4 \equiv b \pmod{p}$.

In the above congruences, $x$ and $y$ and $a$ and $b$ are rational integers. Eisenstein\textsuperscript{12} followed the work of Gauss by defining integers in a field generated by a primitive cube root of unity. He obtained a theorem of cubic reciprocity which provides an algorithm for discovering if an integer $z$ with $z^3 \equiv c \pmod{p}$, where $z$ and $c$ are rational integers, exists. In his proof of the law of cubic reciprocity Eisenstein used formulas from the subject of cyclotomy, which we have already mentioned. Following these investigations, many applications of formulas in cyclotomy, in order to obtain results involving rational integers only, were made by Jacobi, Dirichlet, Cauchy, and others, and some of these will now be discussed. For example, Dirichlet\textsuperscript{12} showed how solutions in rational integers $y(x \neq 0)$ and $x$ may always be obtained for the Pellian equation, $x^2 - ny^2 = 1$, where $n$ is a rational integer $> 1$, making use of some results in cyclotomy.

Jacobi\textsuperscript{14} and Cauchy\textsuperscript{18} obtained the relation (closely related to equation (1))

$$x^2 + \Delta v^2 = p^k,$$

where $k = (\Sigma b - \Sigma a)/\Delta$, $p = \Delta n + 1$, and $\Delta$ is an integer of the form $8m + 7$. Also, $\Sigma a$ stands for the sum of the quadratic residues $a$ of $\Delta$, $0 < a < \Delta$, and $\Sigma b$ is the sum of the quadratic nonresidues $b$, $0 < b < \Delta$. If $\Delta$ is a prime number in the above, the expression

$$\frac{\Sigma b - \Sigma a}{\Delta}$$

is the number of improperly primitive classes of quadratic forms of determinant $-\Delta$. This result was conjectured by Jacobi in advance of Dirichlet’s proof that the statement is true. The Jacobi-Cauchy result above was derived by the use of cyclotomy, and the method was entirely algebraic and different from that employed by Dirichlet in deriving a closed form for the class number of any binary quadratic homogeneous form. The latter method involved the use of analytic expressions for the class number.

If $l$ is an odd prime, Kummer used a primitive $l$th root of unity, say $\zeta$, and defined integers in the field generated by $\zeta$ as being of the form

$$a_0 + a_1 \zeta + \ldots + a_{l-2} \zeta^{l-2},$$

where the $a$’s are rational integers. He developed a theory of such expression which is generally called the theory of cyclotomic fields. He made these investi-
gations in order to be able to generalize the cubic law of reciprocity to the case where we have \( ith \) powers instead of cubes. Also, he wished to make applications to Fermat's last theorem. In efforts to prove this statement, he built up a theory of ideals in such cyclotomic fields. In his work along those lines he found it necessary to use analytic methods in addition to algebraic ones.

So far we have discussed how efforts to solve problems in number theory have led to the founding of the theory of linear substitutions, the theory of certain types of algebraic numbers, and the various developments in cyclotomy. We now come to the next important step in which efforts to solve an arithmetic problem led to important new developments in the theory of matrices. In 1860 H. J. S. Smith attacked a problem of finding criteria that a system of linear congruences, modulo \( m \), with \( m \) an arbitrary integer, has solutions, the coefficients in the congruences being rational integers also. We shall not state what these conditions are, but his result depends on what is sometimes called Smith's normal form of a matrix, namely, that every matrix \( A \) of rank \( r \) with elements in a principal ideal ring \( R \), is equivalent to the diagonal matrix \( [h_1, h_2, \ldots, h_r, 0, \ldots, 0] \), where \( h_i \mid h_{i+1} \). Smith employs this in the case where \( R \) is the ring of rational integers, and \( A \) is a square matrix. It seems probable that this was the first time the concepts of elementary divisors and invariants of a matrix appeared in the mathematical literature.

Dedekind and Kronecker later extended Kummer's notions in order to find the general classical theory of algebraic numbers, namely, the consideration of those fields and rings defined by means of an equation of the \( nth \) degree in one unknown with rational coefficients. This gave mathematicians more tools for obtaining results involving rational integers only.

H. Weber, in his Algebra, seems to have been the first to employ to some extent the theory of abstract Abelian groups to number theory. He called the residue classes modulo \( m \) "Zahlklassen." Hilbert, in his Zahlbericht, refers to a ring of algebraic numbers, this term being employed as we use it today. The concept was used independently by Kronecker and Dedekind under different names.

Around the turn of the century the development of abstract algebra proceeded rapidly, particularly through the work of E. H. Moore, Dickson, Wedderburn, and Steinitz. Moore defined an abstract finite field and proved that any such finite field could be represented by means of a Galois finite field. Dickson made many contributions to this theory. He also developed a theory of linear groups in which the coefficients of the linear forms used were elements in a finite field instead of the field of residue classes, modulo \( p \), as had been employed by C. Jordan. Wedderburn made many important contributions to abstract algebra, in particular to the algebra of matrices. E. Steinitz developed the theory of fields in general.

In 1909 Thue obtained his celebrated theorem that if

\[
a_n x^n + a_{n-1} x^{n-1} y + \ldots + a_1 xy^{n-1} + a_0 y^n = c,
\]

with \( c \) and the \( a \)'s given integers and \( a_n \neq 0, n > 2 \), then equation (2) has only a finite number of integral solutions. The proof was obtained by the use of properties of algebraic numbers.

In 1912, H. S. Vandiver developed a theory of finite rings, said theory having been suggested by generalizations of the theory of residue classes modulo \( m \) where \( m \) is any integer, the latter theory being included in the former.
L. J. Mordell used algebraic numbers and quaternions in a number of valuable papers he has written on Diophantine analysis.

In 1938, James Singer proved the following theorem:\footnote{26} "A sufficient condition that there exist \( m + 1 \) integers, \( d_0, d_1, \ldots, d_m \), having the property that their \( m^2 + m \) differences \( d_i - d_j, i \neq j \); \( i = 0, 1, \ldots, m \), are congruent, modulo \( m^2 + m + 1 \), to the integers \( 1, 2, \ldots, m^2 + m \), in some order, is that \( m \) be a power of a prime."

The proof depended on the idea of the use of the concepts of finite projective geometry and finite fields. The writer does not know of any proof using rational integers only. Singer also gave a generalization of the theorem.\footnote{27} These results, it seems to the writer, exhibit a remarkable relation between the additive and the multiplicative properties of integers, and they have received considerable attention from a number of mathematicians.

A. Hurwitz applied quaternions in order to prove a number of theorems concerning the number of representations of positive integers as the sum of four integral squares.

Dickson\footnote{29} developed arithmetics of various types of algebras, particularly those that were then called hypercomplex numbers and made application to various Diophantine equations.\footnote{30}

Now consider the theory of substitutions in group theory. If we write the substitution

\[
S = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 & 4 \end{pmatrix}
\]

we note that as \( x \) ranges over the complete set of residue modulo 5, \( x^3 \) gives a rearrangement of the \( x \)'s given in \( S \). Under these circumstances \( S \) is said to be an analytic representation\footnote{31} of the substitution. This subject has been investigated by Hermite, Dickson, L. J. Rogers, and others.\footnote{32} Dickson\footnote{33} considered linear forms with variables \( x_1 \ldots x_m \) with indeterminate integral coefficients modulo \( p \), a prime. He then defined invariants and covariants involving such forms and later considered similar forms where the integral coefficients are replaced by elements of finite fields. He made many contributions to this theory as well as to related theories.

Following Eisenstein\footnote{34} there is considerable literature on the application of linear transformation to the theory of representing an integer in the form

\[
f = (a, b, c, d) = ax^3 + 3bx^2y + 3cxy^2 + dy^3
\]

where \( a, b, c, \) and \( d \) are rational integers. It was investigated by Eisenstein, Arndt, Pepin, and others.

The little-known theory of Hermitian forms has also been applied to number theory. Following Hermite\footnote{35} we write

\[
v = x + iy, \quad v_0 = x - iy,
\]

and \( u_0 \) for the conjugate imaginary of \( u \), and we have the binary Hermitian form

\[
f(v, u; v_0, u_0) = A v_0 + B u_0 + B_0 v_0 u + C u u_0,
\]

where \( A \) and \( C \) are real, while \( B, B_0 \) are conjugate imaginaries. Then \( f \) takes
only real values under a linear transformation and $BB - AC$ is an invariant, called the determinant of $f$. Definite, indefinite, and reduced forms are defined, and a theory has been built up which is analogous to that of binary quadratic forms. The subject has applications to the theory of quaternary quadratic forms and has been developed mainly by Hermite, Picard, Bianchi, and Humbert. (Cf. Dickson.)

Vandiver\(^{37}\) (1947) used some results published in his 1912 paper\(^{24}\) on finite algebras (rings) in order to develop concepts and results in the theory of Abelian groups.

In 1922 Hecke published his well-known *Theorie der Algebraischen Zahlen*. Here he seems to have been the first to have employed the theory of groups both finite and infinite in setting up a fairly coherent elementary theory of the arithmetic of the rational integers. He also employed some group theoretic ideas in his chapters on algebraic numbers. H. Hasse\(^{38}\) proceeded further along these lines in utilizing Abelian group theory in a development of elementary number theory and the elements of the theory of algebraic numbers. Vandiver and Milo W. Weaver\(^{39}\) carried some of these ideas further by introducing the elements of the theory of semigroups in setting up a part of elementary number theory.

The Diophantine equation

$$dx^2 + cy^2 = z^n, \quad dc = n (d > 1, c > q),$$

has been examined by a number of mathematicians, employing the theory of quadratic fields, and expressions have been found for all the solutions of it provided in the case where the class number of the quadratic field defined by the $\sqrt{-n}$ is prime to $m$ where $m$ is odd. What appear to be the principal papers on this topic are by Pepin,\(^{40}\) A. Brauer,\(^{41}\) and M. Ward.\(^{42}\)

Criteria that the integer 2 is an $t$th power residue of a rational prime has been considered for small values of $e$ by several writers. Beginning with Gauss, who proved that 2 is a biquadratic residue of a prime $p$ of the form $4n + 1$ if and only if $p$ is representable as $x^2 + 64y^2$, Reusche\(^{43}\) stated that criterion for the octavic character of 2 modulo $p = 8n + 1$. This was proved by A. E. Western.\(^{44}\) A. Cunningham\(^{45}\) discovered the criterion for the 16th power residue of the prime $p$ of the form $16n + 1$, which was later proved by A. Aigner,\(^{46}\) who used the class field theory in his proof. A. L. Whiteman\(^{47}\) employing simpler methods depending on cyclotomy, obtained another proof of the 8th and 16th prime residue characters of 2. The method coincides with that used by Gauss for the proof of the biquadratic character of 2 when it is modified to suit this case. All the criteria just mentioned depend on the possibility of representing $p$ as some quadratic form.

A. A. Albert\(^{48}\) proved that the set of positive ternary forms with integral coefficients, of determinant $d = \gamma^{\delta}$, with $\delta$ square-free, will represent all and only those positive integers $a$ which are not in the form $a = \sigma^2 \sigma$, $\sigma$ square-free, $\sigma = a\delta$, $\alpha = 8n + 7$, $(\alpha, d) = 1$ and the Jacobi symbol $(p/\alpha) = 1$, for all prime factors $p$ of $d$. This result is an elegant generalization of the well-known result that all positive integers which are not of the form $4^t(8n + 7)$ can be expressed as the sum of three squares.

Latin squares are closely related to finite projective geometries and finite fields, so E. T. Parker\(^{49}\) employed the latter to advantage in some of his work on Latin squares.
At the present time it is not always the results recently discovered which are applied to obtain new theorems. The tools used may be quite old. For example, Emma Lehmer\textsuperscript{31} employed a result in cyclotomy obtained by Kummer in 1846 in order to obtain her new results concerning cubic and quartic residues of rational odd primes. These results give direct criteria that certain rational integers are quartic or cubic residues modulo $p$. These theorems were derived independently of the use of the laws of cubic and quartic reciprocity.

Eckford Cohen,\textsuperscript{42} in a number of recent papers, has developed properties of Abelian groups which were obtained by the study of the behavior of arithmetic functions. He also has reversed this process and applied the theory of Abelian groups to obtaining properties of arithmetic functions.

Many articles have been written treating the exponential sums used in cyclotomy in order to obtain results concerning conditional equations in finite fields, which usually give new results concerning rational integers, since the finite field of order $p^n$, $p$ prime, is when $n = 1$, isomorphic to the field of residue classes modulo $p$. L. Carlitz, K. H. Hua, H. H. Mitchell, Vandiver, Whiteman, and others have contributed to these topics.\textsuperscript{53}

As we indicated quite early in this paper, our work here is only beginning. However, I should like, in closing, to indicate what we should be concerned with in future articles which we hope to publish on the same general topic. We recall first that we are confining ourselves in the present paper to the consideration of results involving rational integers only, or results which lead to them. However, we have noted in the past the number of published papers in which theorems regarding algebraic numbers were obtained. Yet they might have been specialized to give new theorems concerning the rational integers only. Few of the authors of these papers signalized this fact, however, and in connection with the present paper we have not gone into this type of investigation.

At the beginning of this paper we stated that we would not consider any of the literature which involved algebraic as well as analytic methods in connection with a particular problem in number theory. We did this in order to avoid the necessity of a detailed examination here of proofs of this character, in order to compare the part of each proof depending on analysis and the part depending on algebra. Of course if we did make an examination of this kind in connection with a particular theorem, it may turn out that the use of modern algebra in the proof might be of great importance and therefore worth discussing in a future paper.

Of course when we use algebraic numbers in order to work out new results concerning rational integers, we usually are employing algebraic extensions of the original rational ring or field. Further, in the case where we employ, as has been done in some investigations, the law of reciprocity between $l$th powers in a cyclotomic field defined by $l$th roots of unity, $l$ being prime, we are going further, as no proof of this law of reciprocity is known which does not involve the use of results in the theory of Kummer fields, which are, of course, extensions of cyclotomic fields.

In view of the above statements, I think it quite likely that methods depending on other types of algebraic extensions will be very useful in obtaining new results in number theory involving rational integers only. As we glimpse in that direction, it is noted that Iwasawa\textsuperscript{64} obtained as a special case of certain theorems derived by
means of a method just mentioned, relations involving class numbers of fields defined by \(l^n\)-th roots of unity, \(n \geq 1\). In fact he finds

\[ e_n = kn + ml^n + c, \]

where \(e_n\) is the highest power of \(l\) dividing the class number of \(C_{n+1}\), which denotes the cyclotomic field of order \(l^n\)-th, \(l\) a prime \(\neq 2\), where \(k = k(L/K), m = m(L/K)\) are defined by a certain extension of \(C_n\), and \(c\) is a suitable integer independent of \(n\).

This result connects up with the known properties of \(C_{n+1}\), namely that there exists a closed explicit form of the value of this class number, and the result derived from this gave criteria that the first factor of said class number be divisible by \(l\).

It is possible that this result may be extended to the case where the modulus is \(l^n, n > 1\) by using similar methods. Proof of this last statement depends on what appear to be entirely different considerations than those involved in the proof of Iwasawa's statement. Often when this sort of thing turns up in mathematics the results are exceptionally interesting, particularly when, after further study, relations between the methods are discerned.

* The work on this paper was supported by the National Science Foundation, Grant 8238, at the University of Texas.  
1 The late J. F. Ritt used to tell us that he regarded number theory as "applied mathematics" because so many different parts of mathematics has been applied to it.  

Such applications have been at least to the theories of semigroups, groups, finite rings and fields, and semirings.

4 We have an extensive branch of number theory called the "geometry of numbers," and of course numerous applications have been made to number theoretic problems which are not always regarded as belonging to the geometry of numbers, but geometric methods are used. Here we shall cite such a problem, as considerable literature has grown up around it. The first mention of a special case of it seems to have been by O. Veblen (Am. Math. Monthly, 13, 46 (1906), reprinted by Johnson Reprint Corp., New York,) who proposed it to the readers of the Monthly. He stated the result for \(m^2 + m + 1 = 43\). Cf. Singer for definition of \(m\).

5 Since we are not going to discuss in detail the application of number theory to analysis or the reverse, we shall refer the reader to Hilbert in his introduction to his Zahlbericht of 1897 (Werke, bd. 1, 63-68), who discussed the applications of the theories of algebraic numbers to analysis as well as the applications of analysis to the theory of algebraic numbers, which had been made prior to 1897. Of course, if any mathematician treated in detail the advances which have been made along these lines since, the work would be immense. It would include also applications of abstract algebra, included in the theory of algebraic numbers, to analysis and then conversely.


7 Lagrange (Abh. Acad. Berlin, for years 1770 and 1771) introduced the "Lagrange Resolvent"; Gauss, Disquisitiones Arithmeticae, Sec. 7.

8 Gauss, "Theoria residuorum biquadraticorum, commentatio prima," Werke 2, Art. 23. This is a very unusual type of relation in number theory since binomial coefficients and related arithmetic functions are usually very complicated from the standpoint of divisibility.


14 "Sur la manière de resoudre l'équation $t^4 - pu^4 = 1$ au moyen des fonctions circulaires," Werke, 1, 343.

15 J. für die Mathematik, 9, 189 (1826).

16 Oeuvres (1), nearly all of vol. 3.

17 The applications of the theory of algebraic numbers to Fermat's last theorem and closely related equations are so well-known that they will not be discussed in this paper.


19 Second edition, Braunschweig, 1899. Galois considered a prime $p$ and a polynomial in an indeterminate $x$ of degree $n$ with given integral coefficients, say $F(x)$, which is also irreducible modulo $p$. He then spoke of a "root" of this congruence. If $a$ is one such, then the set,

$$a + a_1a^2 + a_2a + \ldots + a_{n-1}a^{n-1}$$

with the $a$'s ranging over 0, 1, ..., $p - 1$, independently, give $p^n$ distinct elements. Weber (p. 306) notes that such elements form a finite field of order $p^n$.

20 Werke, 1, 121 (1932).


22 Linear Groups, B. J. Teubner, Leipzig (1901).


28 Ibid., 385.


30 Algebras and Their Arithmetics (The University of Chicago Press, 1923).

31 Ibid., 198.


33 Ibid., 290-291.

34 Ibid., chap. 19.


36 Ibid., 269.

37 Ibid., chap. 15.


41 "Sur certains nombres complexes de la forme $a + b\sqrt{-c}$," J. Math., 3er serie, Bd 1, 317-372 (1875).
HOMOLOGY THEORIES AND DUALITY*

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Communicated by Saunders Mac Lane, February 8, 1960

1. It has been known for some years that the integral cohomology group $H^n(X; \Pi)$ of a (reasonable) space $X$ with coefficients in an abelian group $\Pi$ can be characterized as the group of homotopy classes of maps of $X$ into a certain space $K(\Pi, n)$. The spaces $K(\Pi, n)(n = 0, 1, 2, \ldots)$ are related by the fact that each is the loop-space of the next; and it is this property that is crucial in setting up the apparatus of cohomology theory. In fact it is known that, if $E = \{E_n\}$ is a sequence of spaces, each the loop-space of the next, then the groups $H^n(X; E) = [X; E_n]$ satisfy all of the Eilenberg-Steenrod axioms for a (reduced) cohomology theory except the dimension axiom.

It is natural to ask whether there are corresponding homology theories. Of course, if a cohomology theory is given one can define homology groups (at least for a finite complex $X$) as the cohomology groups, suitably reindexed, of the complement of $X$ in a sphere of sufficiently large dimension. This definition is awkward to work with and has the disadvantage of failing to be intrinsic. The object of this note is to develop an intrinsic theory of generalized homology groups, and to prove...