CLASSES OF HOLOMORPHIC FUNCTIONS OF SEVERAL VARIABLES IN CIRCULAR DOMAINS*

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We will extend a theorem of (F. and M. Riesz and) Hardy-Littlewood from holomorphic functions \( f(z) \) in the unit disk \( |z| < 1 \) to holomorphic functions of several variables \( f(z_1, \ldots, z_k) \) in general circular domains.

**Theorem 1.** If \( f(z) \) is holomorphic in \( |z| < 1 \) and, for some \( \lambda > 0 \),

\[
\sup_{0 < r < 1} \int_0^1 |f(re^{2\pi i\theta})|^\lambda d\theta \equiv C^\lambda < \infty,
\]

then

\[
\int_0^1 \sup_{0 < r < 1} |f(re^{2\pi i\theta})|^\lambda d\theta \leq \alpha_\lambda C^\lambda,
\]

where \( \alpha_\lambda \) depends only on \( \lambda \) and nothing else.

Also, for \( r_1, r_2 < 1 \),

\[
\int_0^1 |f(re^{2\pi i\theta}) - f(ve^{2\pi i\theta})|^\lambda d\theta \to 0 \quad \text{as} \quad (r_1, r_2) \to (1, 1),
\]

and there exists a measurable function \( F(\theta) \) on the boundary \( \zeta = e^{2\pi i\theta} \) such that, for \( 0 < r < 1 \),

\[
\int_0^1 |f(re^{2\pi i\theta}) - F(\theta)|^\lambda d\theta \to 0 \quad \text{as} \quad r \to 1.
\]

The quantity

\[ M(r) = \int_0^1 |f(re^{2\pi i\theta})|^\lambda d\theta \]

is monotonely increasing in \( r \). Using this, it is not hard to verify that Theorem 1 gives rise to the following generalization of itself.

**Theorem 2.** If our holomorphic function depends on a parameter \( \xi \), \( f(z) = \varphi(z; \xi) \); if \( \xi \) is the point of a Lebesgue measure space \( \{ X: (\xi), d\mu(\xi) \} \) and \( \varphi(z; \xi) \) is measurable in \( (z; \xi) \); if for almost all \( \xi \), \( \varphi(z; \xi) \) is holomorphic in \( |z| < 1 \); and if, for \( \lambda > 0 \),

\[
\sup_{0 < r < 1} \int_X d\xi \int_0^1 |\varphi(re^{2\pi i\theta}; \xi)|^\lambda d\theta \equiv C^\lambda < \infty,
\]

then, for the same \( \alpha_\lambda \),

\[
\int_X d\xi \int_0^1 \sup_{0 < r < 1} |\varphi(re^{2\pi i\theta}; \xi)|^\lambda d\theta \leq \alpha_\lambda C^\lambda;
\]

also,

\[
\int_X d\xi \int_0^1 |\varphi(re^{2\pi i\theta}; \xi) - \varphi(ve^{2\pi i\theta}; \xi)|^\lambda d\theta \to 0 \quad \text{as} \quad (r_1, r_2) \to (1, 1)
\]

and there exists a function \( \Phi(\theta; \xi) \) measurable in \( (\theta; \xi) \) such that

\[
\int_X d\xi \int_0^1 |\varphi(re^{2\pi i\theta}; \xi) - \Phi(\theta; \xi)|^\lambda d\theta \to 0.
\]
Now, in the space of $k$ complex variables $(z_1, \ldots, z_k)$ let $D$ be a (bounded) circular domain containing the origin. Thus, if $(a_1, \ldots, a_k) \in D$, then also $(za_1, \ldots, za_k) \in D$ for $|z| \leq 1$. Let $B$ denote the boundary of $B$ with relative topology. We take on $B$ any finite Borel measure $\omega(B_0)$, $B_0 \subset B$, subject only to the condition that it be circularly invariant. By this we mean that if we replace all points $(a_1, \ldots, a_k)$ of $B_0$ by $(e^{2\pi i} a_1, \ldots, e^{2\pi i} a_k)$ and denote the new set by $B_0'$ then $\omega(B_0') = \omega(B_0)$. The leading feature of such a measure is the following. If $\Psi(\xi_1, \ldots, \xi_k)$ is a (non-holomorphic) function on $B$ summable with respect to $\omega(B_0)$ then

$$\int_B \Psi(\xi_1, \ldots, \xi_k) d\omega = \int_0^1 d\theta \int_B \Psi(e^{2\pi i} \xi_1, \ldots, e^{2\pi i} \xi_k) d\omega.$$ 

Because of this, Theorem 2 implies as follows.

**Theorem 3.** If $f(z_1, \ldots, z_k)$ is holomorphic in a bounded circular domain $D$ with origin, and if

$$\sup_{0 < r < 1} \int_B |f(r\xi_1, \ldots, r\xi_k)|^\lambda d\omega(\xi) \equiv C^\lambda < \infty,$$

$\lambda > 0$, relative to a finite circularly invariant Borel measure $\omega(B_0)$ on the boundary $B$ of $D$; then

$$\int_0^1 \sup_{0 < r < 1} |f(r\xi_1, \ldots, r\xi_k)|^\lambda d\omega(\xi) \leq a_\lambda C^\lambda.$$ 

Also

$$\int_B |f(r\xi_1) - f(r\eta_1)|^\lambda d\omega(\xi) \to 0 \text{ as } (r_1, r_2) \to (1, 1)$$

and there exists a function $F(\xi) \equiv F(\xi_1, \ldots, \xi_k)$ on $B$, measurable relative to $\omega(B_0)$ such that

$$\int_B |f(r\xi_1) - F(\xi)|^\lambda d\omega(\xi) \to 0 \text{ as } r \to 1.$$

Theorem 3 applies in particular to the sphere

$$S_k: \ |z_1|^2 + \ldots + |z_k|^2 < 1$$

with “ordinary” $$(2k-1)$$-dimensional surface area $\omega(B_0)$; and it also applies to the polyecylinder

$$P_k: \ |z_1| < 1, \ldots, |z_k| < 1,$$

if the measure $\omega(B_0)$ on its boundary $B$ is defined in such a manner that

$$\int_B \Psi(\xi_1, \ldots, \xi_k) d\omega(\xi) = \int_0^1 \ldots \int_0^1 \Psi(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_k}) d\theta_1 \ldots d\theta_k.$$ 

This measure is concentrated entirely on the $k$-dimensional subset $|\xi_1| = 1, \ldots, |\xi_k| = 1$ of the total $(2k-1)$-dimensional boundary, but it is a finite Borel measure and circularly invariant, and this was all we have required of it.

We note the corollary ($\lambda = 1$) that if $f(z_1, \ldots, z_k)$ is holomorphic in $P_k$ and

$$\sup_{0 < r < 1} \int_0^1 \ldots \int_0^1 |f(re^{2\pi i \theta_1}, \ldots, re^{2\pi i \theta_k})| d\theta_1 \ldots d\theta_k < \infty$$

then there exists a multiperiodic function $F(\theta) \equiv F(\theta_1, \ldots, \theta_k)$ of class $L_1$ such that

$$\int_0^1 \ldots \int_0^1 |f(r\theta_j) - F(\theta_j)| d\theta_1 \ldots d\theta_k \to 0, r \to 1.$$
This non-trivial proposition was originally stated and proved by this author, and the proof employed the radial maximum of Hardy-Littlewood, but in a much more complicated manner than the present approach suggests. Also an entirely different proof of this "corollary" was recently given by Helson and Lowdenslager, and we wish to point out that our original proof and their recent method both yield a somewhat stronger result which cannot be easily stated within the present set-up.

Finally we observe that for \( \lambda > 1 \) we have the following theorem again obtainable from its one variable prototype.

**Theorem 4.** For \( \lambda > 1 \) all parts of Theorem 3 (and also of Theorem 2) are valid if instead of a holomorphic function in \( D \) we consider a function \( u(z_1, \ldots, z_k) \) which is the real part of a holomorphic function.

Also, for \( \lambda = 1 \) there remains the statement that

\[
\sup_{0 < r < 1} \int_B |u(r\xi_j)| \log^+ |u(r\xi_j)| \, d\omega(\xi) \leq C < \infty
\]

implies the relation

\[
\int_B \sup_{0 < r < 1} |u(r\xi_j)| \, d\omega(\xi) \leq \alpha C + \beta
\]

and the existence of a function \( U(\xi_j) \) for which

\[
\int_B |u(r\xi_j) - U(\xi_j)| \, d\omega(\xi) \to 0, \; r \to 1.
\]

**Remark:** Note that the log factor in the assumption appears in first power for all complex dimensions \( k \).

One can also extend a theorem of M. Riesz on conjugate functions and the statement is as follows.

**Theorem 5.** If \( f(z_j) = u(z_j) + iv(z_j) \) is holomorphic in \( D \) and \( v(0) = 0 \), then, for \( p > 1 \),

\[
\sup_r \int_B |v(r\xi_j)|^p \, d\omega \leq \gamma_p \sup_r \int_B |u(r\xi_j)|^p \, d\omega
\]

and, for \( p = 1 \),

\[
\sup_r \int_B |v(r\xi_j)| \, d\omega \leq \gamma \sup_r \int_B |u(r\xi_j)| \log^+ |u| \, d\omega + \delta,
\]

where \( \gamma_p, \gamma, \delta \) are fixed constants.

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