Steenrod\(^1\) has defined the cyclic reduced powers

\[ P^j: H^m(X; Z_p) \to H^{m+2j(p-1)}(X; Z_p) \] (1)

for any topological space \(X\) and nonnegative integer \(j\). The \(P^j\) are linear mappings, stable under suspension and natural in \(X\). Furthermore, the algebra \(A\) of all stable cohomology operations over \(Z_p\) is generated by linear combinations of the compositions of \(P^j\) and \(\Delta\), where the Bockstein

\[ \Delta: H^m(X; Z_p) \to H^{m+1}(X; Z_p) \] (2)

is associated with the exact coefficient sequence

\[ 0 \to Z_p \to Z_{p^2} \to Z_p \to 0. \]

Adem\(^2\) and Cartan\(^3\) have studied relations between products of \(P^j\)'s and \(\Delta\)'s. It turns out that if \(j\) is not a power of \(p\), \(P^j\) can be expressed as a linear combination of compositions \(P^{i_1} \cdots P^{i_n}\), where \(i_v < j\), \(v = 1, \ldots, n\). That is, in the Steenrod algebra \(A\) the elements \(P^j, j \neq p^r\), are decomposable.\(^2\)

Associated with the Steenrod algebra \(A\) over \(Z_p\) are certain stable secondary cohomology operations.\(^4\) We prove that there exist stable secondary cohomology operations \(\Theta, \Psi_k (k = 1, 2, \ldots)\) such that \(\Theta\) is defined on classes \(u \in H^m(X; Z_p)\) with \(\Delta u = 0, P^1 u = 0\), and \(\Theta(u)\) is a coset of the group

\[ P^2 H^m(X; Z_p) + \left(\frac{1}{2} \Delta P^1 - P^1 \Delta\right) H^m + 2p^{-2} (X; Z_p) \] (3)

in \(H^{m+4(p-1)}(X; Z_p)\); the operations \(\Psi_k (k > 0)\) are defined on classes \(v \in H^m(X; Z_p)\).
$Z_p$) such that $\Delta v = 0$, $P^{p^i}v = 0$, $(i = 0, 1, \ldots, k)$, and $\Psi_k(v)$ is a coset of

\[
P^{p^i}H^m(X; Z_p) + \sum_{i=0}^k \partial_i H^{m-1+2p^i(p-1)}(X; Z_p)
\]

in the group $H^{m+2p^i(p-1)}(X; Z_p)$, where $\partial_i$ are homogeneous elements of the Steenrod algebra $A$ with odd gradings.

**Theorem 1.** Let $k$ be a nonnegative integer. There exists a constant $v_k$, nonzero in $Z_p$, elements $a_{k,i}$, $b_{k,i}$, $c_{k,i}$ of positive grading in the Steenrod algebra, and secondary cohomology operations $\Gamma_\gamma$ of odd degree, such that

\[
\{ v_k p^{i+1} \} = \sum_{i=1}^k a_{k,i} \Psi_i + b_{k,i} \Gamma_{\gamma} + \sum_{\gamma} c_{k,i} \Gamma_{\gamma},
\]

the relation holding modulo the total indeterminacy of the right-hand side.

That such a relation exists, with the scalar $v_k$ possibly zero, is an immediate consequence by the methods of J. F. Adams\(^4\) of the following theorem.

**Theorem 2.** The vector spaces $\text{Ext}_A^i(Z_p, Z_p)$ for $i = 1, 2$ have the following $Z_p$ bases:

1. $\text{Ext}_A^1(Z_p, Z_p)$ has as basis certain classes $h_i$ $(i = 0, 1, \ldots)$ of grading $2p^i(p - 1)$, $a_0$ of grading 1;
2. a $Z_p$ basis for $\text{Ext}_A^i(Z_p, Z_p)$ is furnished by certain classes $h_i h_{j}$, $i < j - 1$ $(i = 0, 1, \ldots, j = 2, 3, \ldots)$ of grading $2(p - 1)(p^i + p^j)$, $h_{i} a_0 (i \neq 0)$ of grading $2p^i(p - 1) + 1$, $\mu_i$ of grading $2(p - 1)(p^i + 1) + 1$, $\nu_i$ of grading $2(p - 1)(2p^i + 1) + p^i$, $\lambda_i$ of grading $2(p - 1)(p^i + 1)$, $\rho$ of grading $4(p - 1) + 1$, and $a_0 a_0$ of grading $2$;
3. the elements $\lambda_0 \alpha$ in $\text{Ext}_A^3(Z_p, Z_p)$ of grading $2p^i + 1(p - 1) + 1$ are nonzero.

The theorem is proved by using the Adams spectral sequence for Hopf algebras. The proof is made easier by introducing Steenrod operations. These operations are defined on $\text{Ext}_G^i(\ast(Z_p, Z_p))$, where $G$ is a graded, connected Hopf algebra over $Z_p$ with associative product and associative and commutative diagonal:

\[
P^1: \text{Ext}_G^i(\ast(Z_p, Z_p)) \rightarrow \text{Ext}_G^{i+2/(p-1), p^1}(Z_p, Z_p),
\]

\[
\Delta: \text{Ext}_G^{i+1}(Z_p, Z_p) \rightarrow \text{Ext}_G^{i+1, p^1}(Z_p, Z_p).
\]

The operations have most of the properties of the usual Steenrod cohomology operations. In particular, the Adem relations\(^3\) and the Cartan formula for cup products\(^4\) are valid. A departure from the topological case appears in the shift of the second grading. $P^0$ is no longer the identity operation:

\[
P^0 h_i = h_{i+1},
\]

\[
\Delta h_i = \lambda_i.
\]

Let $\alpha H^2(Z, 2; Z_p)$ be the fundamental cohomology class. The indeterminacies involved in (5) vanish for $H^*(Z, 2; Z_p)$, thus the relation becomes a relation on classes, not cosets. We find the unknown coefficient $v_k$ in Theorem 1 by evaluating the relation (5) on the class $\alpha^{p^{i+1}}$. For this, we need to know the values of $\partial$ and $\Psi_k$ on $\alpha^{p^{i+1}}$.

**Theorem 3.** Let $t$ be a positive integer. There exist nonzero constants $b$, $c_k$ in $Z_p$ such that

\[
\partial(b \alpha^{p^{i+1}}) = b \alpha^{p^{i+2/(p-1)}}.
\]
\[ \Psi_k(\alpha^{m+1}) = c_d \alpha^{m+1} + p^{k(d-1)}. \]  

(9)

The theorem is proved by considering the cohomology of universal example spaces associated with a minimal resolution of \( Z_p \) over \( A \). Since these spaces can be taken to be iterated loop spaces, both the homology and cohomology are Hopf algebras over \( Z_p \) with commutative associative product and diagonal. The universal examples for \( 2p^* + 1 \)-dimensional classes turn out to be homotopically equivalent to Cartesian products of \( K(Z_p, n)'s \), but not as loop spaces. In cohomology this is reflected by the decomposition of the cohomology into a tensor product as algebras, but not as coalgebras. It turns out that primitive elements defining the operations \( \partial, \Psi_k \) can be completely described, enabling us to evaluate \( \Psi_k(\alpha^{m+1}) \) and \( \partial(\alpha^p) \) (a nonzero integer) are then found by using the Adams formula for secondary operations on cup products.4

At a critical point in the proof, a consequence of a theorem by W. Browder\(^6\) is used strongly. Let

\[ \alpha_{m+n}: H_n(\Omega X; Z_p) \to H_{n+1}(X; Z_p) \]

be the homology suspension.

**Theorem 4.** If \( X \) is an \( H \)-space, then Kernel \( \alpha_{m+n} \) is precisely the set of decomposable elements of \( H_n(\Omega X; Z_p) \) if \( n \) cannot be written in the form \((2m)p' - 2\), where \( f \) and \( m \) are integers.

Theorem 1 is related to the question of the existence of elements of mod \( p \) Hopf invariant one. The mod \( p \) Hopf invariant is a homomorphism

\[ \gamma_{p,i}: \Pi_{m+n}(S^m) \to Z_p, \]  

(10)

where \( n = 2i(p-1) - 1 \), defined as follows: given a class \( \alpha \) in \( \Pi_{m+n}(S^m) \), pick an element \( f \) of \( \alpha \). Define the cell complex \( K_f = S^m \cup_f E^{m+n} \) by attaching an \((m+n)\)-cell to \( S^m \) by \( f \). Pick standard generators \( \iota'_0, \iota'_m, \iota'_{m+n} \) for the three nonzero groups of \( H^*(K_f; Z_p) \). Then \( \gamma_{p,i}(\alpha) \) is defined by

\[ P^{i} \iota'_{m+n} = \gamma_{p,i}(\alpha) \iota'_{m+n}. \]  

(11)

Adem\(^2\) proved that \( \gamma_{p,i} \) is zero, unless \( i = p' \), for some nonnegative integer \( r \). Borel and Serre\(^7\) and Toda\(^8\) proved that \( \gamma_{p,1} \) is nontrivial. Milnor\(^9\) showed that \( \gamma_{p,1} \) is zero on the image of the Whitehead \( J \)-homomorphism if \( i > 1 \). These results are contained in the following immediate consequence of Theorem 1.

**Theorem 5.** If \( i > 1 \), then \( \gamma_{p,i} \) is identically zero.

An immediate result of this (for \( p = 3 \)) is \( 8^{10} \) the following theorem.

**Theorem 6.** Let \( \iota_{2n} \) be the homotopy class of the identity map in \( \Pi_{2n}(S^{3n}) \), and let

\[ [\iota_{2n}, [\iota_{2n}, \iota_{2n}]] \]  

(12)

be the iterated Whitehead product. Then if \( n > 1 \), the element \([\iota_{2n}, [\iota_{2n}, \iota_{2n}]]\) is an element of order 3 in \( \Pi_{6n-2}(S^{3n}) \).

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1 Steenrod, N. E., these **Proceedings**, 39, 213-217 (1953).
NOTE ON POLYNOMIAL APPROXIMATION ON A JORDAN ARC*

BY J. L. WALSH

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY

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In two recent notes1, 2 the present writer has discussed the invariance of degree of polynomial and trigonometric approximation under change of independent variable. However, those notes did not emphasize approximation in the complex plane on a Jordan arc rather than on a Jordan curve; the object of the present note is to indicate rapidly additional results, concerning Jordan arcs.

We state for reference

THEOREM 1. Let $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of positive numbers approaching zero, where we suppose $\varepsilon_{[n/\lambda]} = O(\varepsilon_n)$ for every positive integral $\lambda$ (here $[m]$ denotes the largest integer not greater than $m$), and where for every $r, 0 < r < 1$, we have $r^n = O(\varepsilon_n)$. Let $E$ be a Jordan arc in the $z$-plane, and let $D$ be a region containing $E$. Let the function $f(z)$ be defined on $E$, let the functions $f_n(z)$ be analytic in $D$, and suppose $(n = 1, 2, 3, \ldots)$

\begin{align}
|f(z) - f_n(z)| & \leq A_1\varepsilon_n, \quad z \text{ on } E, \\
|f_n(z)| & \leq A_2R^n, \quad z \text{ in } D.
\end{align}

Then there exist polynomials $p_n(z)$ in $z$ of respective degrees $n$ such that

\begin{equation}
|f(z) - p_n(z)| \leq A_3\varepsilon_n, \quad z \text{ on } E.
\end{equation}

Here and below, the letter $A$ with subscript denotes a positive constant independent of $n$ and $z$.

Theorem 1 is contained in the comments to the Corollary of Theorem 1 in reference 1. Indeed, it is not necessary that $E$ be a Jordan arc; it is sufficient if $E$ is a closed bounded set.

If $E, D,$ and $f(z)$ satisfy the hypothesis of Theorem 1 (not including inequalities (1) and (2)), if $D$ is bounded, and if inequality (3) is valid for polynomials $p_n(z)$ of respective degrees $n$, then inequality (2) with $f_n(z)$ replaced by $p_n(z)$ is also valid if $R$ is suitably chosen. That is to say, if $D$ is bounded, inequality (3) is not merely necessary but also sufficient for inequalities (1) and (2). Indeed, inequality (3) implies $|p_n(z)| \leq A_4, z \text{ on } E$, which by the generalized Bernstein Lemma (§4.6 of reference 3) implies $|p_n(z)| \leq A_4R^n, z \text{ on } E_R$, where $E_R$ is the level locus $g(z) = \log R \ (>0)$ of Green’s function $g(z)$ for the complement of $E$ with pole at infinity. For sufficiently large $R$, $D$ lies interior to $E_R$.

The particular interest of Theorem 1 lies largely in the fact that the hypothesis