CONVEX TYPE VARIETIES*,†

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1. Introduction.—Strictly convex hypersurfaces in affine $n$-space, $n > 1$, meet every straight line in at most two points, and lie on one side of every tangent hyperplane. Strictly comonotone curves meet every hyperplane in at most $n$ points, and lie for even $n$ on one side of every osculating hyperplane. Having thus singled out varieties of dimensions $n - 1$ and 1 with to some extent analogous properties we are led to ask whether there are similar varieties $V$ of a dimension $m$ between 1 and $n - 1$. We assume differentiability as needed.

We shall see that the answer to the question posed is negative for "exact minimal order", affirmative for "unilaterality" (as defined below). But while examples of strictly convex hypersurfaces (e.g., the sphere) were evident as soon as $n$-space was considered (for $n = 2$ and 3 coinitially with geometry), and while a special strictly comonotone curve, the norm curve, is maybe the simplest algebraic space curve,† the unilateral varieties other than curves and hypersurfaces perhaps escaped detection because of their non-existence for $n < 10$.

2. Notations and Definitions.—The desired properties of a variety of dimension $m$ in $n$-space are:

$M_s$ (exact minimal order): $V$ intersects every $(n - m)$-flat (linear variety of dimension $n - m$) in at most $n - m + 1$ points.

$U$ (unilaterality): for any point $x$ on $V$, $V - \{x\}$ lies in an open halfspace bounded by a hyperplane $H_x$ that is the flat of highest contact at $x$.

Weaker related properties also considered:

$M$ (minimal order): $V$ intersects almost every $(n - m)$-flat (i.e. all but a set of measure 0) in at most $n - m + 1$ points.

$U_l$ (local unilaterality): for every $x$ on $V$ there exists a neighborhood $N$ of $x$ such that $v \cap N - \{x\}$ lies in an open halfspace bounded by $H_x$.

$F$ (flexion): for any $x$ on $V$, $V \cap H_x = \{x\}$.

$F_l$ (local flexion): for any $x$ on $V$ there exists an $N$ such that $V \cap N \cap H_x = \{x\}$.

Further $B$ and $S$ denote boundedness and simplicity (non-self-intersection) in case of varieties without a lower-dimensional boundary.

Unilaterality is a pronounced case of outwardness, i.e., the property of a variety consisting of the extreme points of its convex hull.
3. Results. — We shall prove

Theorem 1. For every variety with property $M_\epsilon$, $m = 1$ or $m = n - 1$.

Theorem 2. There are varieties with property $U_1$ if and only if $n = \left(\frac{m+k}{m}\right)$, where $k$ is odd. Moreover for these $n$ and $m$ there exist varieties with properties $U_1$, $B$, and $S$.

Theorem 3. There exist varieties with property $F_1$ and without property $U_1$ if and only if $m = 1$ and $n$ is odd. None of these varieties has property $B$; all have properties $F_1$ and $S$.

We also give examples of varieties with property $M$ and $m \neq 1, n - 1$.

4. Proofs. — Proof of Theorem 1: Property $M_\epsilon$ is obviously inherited by the intersections of $V$ with flats; $n - m$ remains unchanged. Hence it suffices to prove that $m = 2$ and $M_\epsilon$ imply $n = 3$.

For $m = 2$ and even $n$ any $n$ points on $V$ determine a hyperplane $H$, hence a comonotone curve $V \cap H$ and on it a cyclic ordering of the $n$ points. The ordering is preserved when the points move continuously on $V$, as long as they remain distinct; such a move can be performed so that two of the points exchange position while the others remain fixed (the points may be assumed near each other on a connected part of $V$). For $n > 3$ this would be a new ordering, again a contradiction.

(This case of the proof can also be extended to even $n$.)

Proof of Theorem 2: (1) The tangent flats of $V$ at a general point have dimension $m, \left(\frac{m+2}{2}\right) - 1, \ldots, \left(\frac{m+k}{k}\right) - 1 < n$. The last of these is a hyperplane $H_x$ only if $n = \left(\frac{m+k}{k}\right)$. If near $x = (x_1, \ldots, x_n)$ the parametric expansion of $V$ is

$$x_1 + \sum_{j=1}^{\infty} c_{ij}u^i; i = 1, \ldots, n,$$

$$j = (j_1, \ldots, j_m), u^i = u_1^{j_1} \ldots u_m^{j_m}, J = j_1 + \ldots + j_m,$$

then the $c_{ij}$ with $1 \leq J \leq k$ determine the direction $(d_1, \ldots, d_k)$ orthogonal to $H_x$, and the distance of a point of $V$ near $x$ (and on the same branch) from $H_x$ is $\Phi^{+}$ higher terms, $\Phi = \sum_{j=k+1} \Delta c_{ij}u^i$. Property $U_1$ means that the form $\Phi$ of order $k + 1$ (which cannot vanish identically for all $x$) is definite and hence $k$ odd. Observe that $\Phi$ is a constant times the determinant formed by the $c_{ij}, 1 \leq J \leq k$, and the $\sum_{j=k+1} c_{ij}u^i$.

(2) If the $x_i, i = 1, \ldots, n - 1$ are the different products $u^i$ with $1 \leq J \leq k$ and if $x_1$ is a definite form $\Psi(u) = \sum_{j=k+1} c_{i}u^i$, then for every parameter value system $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_m)$ we have $x_1 = (u - \tilde{u})^i + 1.d.t.$ (terms of lower degree in $u - \tilde{u}$). $x_2 = \Psi(u - \tilde{u}) + 1.d.t.$ The above determinant, taken at $\tilde{x}$, equals $\Psi(u - \tilde{u})$; hence the rational variety $V$ defined by the $x_i$ has properties $U$ and $S$. Moving the tangent hyperplane $H$ slightly we obtain a hyperplane $H'$ disjoint from $V$; a projective transformation such that $H'$ becomes the hyperplane at infinity makes $V$ bounded.
Proof of Theorem 3: Here $\Phi$ is not definite, while $\Phi = 0$ for no real $(u_0, \ldots, u_n) \neq (0, \ldots, 0)$. This is equivalent to $m = 1, k + 1 = n$ odd. The $V$ are locally comonotone; if comonotone they have property $F$. A closed bounded curve crosses every hyperplane in an even number of points; for odd $n$ the curve crosses some hyperplanes in at least $n + 1$ points; and between $n + 1$ crossings of a hyperplane and a curve there is always a point of hyperosculation for which $F_i$ does not hold. 2

5. Remarks.—Note that every definite form $\Psi(u)$ gives rise to a unilateral rational variety. For $m = 1$ we obtain the norm curves, for $m = n - 1$ the strictly convex quadrics. The unilateral rational varieties belonging to definite forms $\Psi(u)$ are easily seen to be intersections of quadrics.

A unilateral variety with $m < n - 1$ on a strictly convex quadric (e.g., a norm curve for even $n$) is, by projection of the quadric to a hyperplane and inversion, transformed into a variety in $(n - 1)$-space that lies on one side of every osculating hyperplane.

Examples of rational varieties with property $M$, $m = 2$, $n > 3$ are given by $(u, v, \sqrt{u^2}, \sqrt{w})$ and $(u, v, \sqrt{u^2}, \sqrt{w}, \sqrt{v^2})$.

For even $n$ and $m = 1$ property $U_1$ without $U$ can hold with $B$ and $S$, with either, or neither; while $U$ implies $S$ but not $B$.

The numbers $n < 2^7$ for which $m$-dimensional unilateral varieties with $m \neq 1$ $n - 1$ exist are by Theorem 2:

$m = 2$: 10, 21, 36, 55, 78, 105

$m = 3$: 20, 56, 120

$m = 4$: 35

$m = 5$: 56

$m = 6$: 84

$m = 7$: 120

Since $n \geq \left(\frac{m + 3}{3}\right)$, we have $m < (6n)^{1/3}$. The number of numbers $n < n_0$ is asymptotically equal to $(n_0/2)^{1/2}$.

The convex hull of finitely many points on a unilateral variety other than a convex hypersurface or comonotone curve is a unilateral polyhedron. Also the corresponding cones and the dual varieties, cones and polyhedra, as well as the dissections of space by finitely many flats $H_z$ of highest contact with a unilateral variety deserve attention.

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