THE PATHS OF RAYS OF LIGHT IN GENERAL RELATIVITY

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1. This paper deals with a space $V_4$ of four dimensions which admits minimal geodesics as paths of rays of lights, that is curves for which

$$ g_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = 0 \quad (1) $$

This means that the symmetric metric tensor $g_{ij}$ is not positive definite.

The equation of geodesics in $V_4$ is

$$ \frac{d^2x^i}{ds^2} + \left\{ \begin{array}{c} \frac{dx^j}{ds} \\ \frac{dx^k}{ds} \end{array} \right\} \frac{dx^i}{ds} = 0 \quad (2) $$

The second term in this equation stands for the sum of terms as $j$ and $k$ take the values 1 to 4. This convention is used throughout this paper, namely that when in a term the same letter enters as a subscript and superscript, it means that the one term stands for the sum of terms as the index takes the values 1 to 4.

We put

$$ \frac{dx^i}{ds} = \lambda^i \quad (3) $$

where $\lambda^i$ are the contravariant components of a vector, in terms of which equation (2) of geodesics is

$$ \left( \frac{\partial \lambda^i}{\partial x^j} + \left\{ \begin{array}{c} i \\ j \end{array} \right\} \lambda^k \right) \lambda^j = 0 $$

which we write in the form

$$ \lambda^i_j \lambda^j = 0 \quad (4) $$

where

$$ \lambda^i_j = \frac{\partial \lambda^i}{\partial x^j} + \lambda^k \left\{ \begin{array}{c} i \\ j \end{array} \right\} $$

which is the covariant derivative of $\lambda^i$ with respect to $x^j$.2

Here, and throughout this paper, a component of a vector or other tensor followed by a comma and an index denotes the covariant derivative of the quantity with respect to $x$ with this index.

2. The covariant components $\lambda_i$ of the vector $x^i$ are given by

$$ \lambda_i = g_{ik} \lambda^k \quad (5) $$

Since the covariant derivative of $g_{ik}$ is equal to zero,3 when equation (4) is multiplied by $g_{ik}$ and summed for $i$, the result is

$$ g_{ik} \lambda_i \lambda^j = (g_{ik} \lambda^j)_j \lambda^j = \lambda_k \lambda^j = 0 \quad (6a) $$

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which when \( k \) is replaced by \( i \) becomes

\[
\lambda_{ij} \lambda^j = 0
\]  
(7)

where

\[
\lambda_{ij} = \frac{\partial \lambda_j}{\partial x^i} - \lambda_k \left\{ \frac{k}{ij} \right\}
\]  
(8)

which is the covariant derivative of \( \lambda_i \) with respect to \( x^i \).

By means of equation (3) the equation (1) becomes

\[
g_{ij} \lambda^j \lambda^i = 0
\]  
(9)

which in accordance with equation (6) becomes

\[
\lambda^i \lambda_i = 0.
\]  
(10)

3. When equation (10) is differentiated covariantly with respect to \( x^i \) the result is

\[
\lambda_i \lambda^i + \lambda^i \lambda_j = 0.
\]

Since the covariant derivative of \( g^{ik} \) is equal to zero, we have

\[
\lambda^i \lambda_j = \lambda_i (g^{ik} \lambda_k)_j = \lambda_i g^{ik} \lambda_k,j = \lambda^i \lambda_k,j.
\]

Hence the above equation is equal to twice the equation

\[
\lambda_i \lambda^i = 0.
\]  
(11)

In accordance with equation (10), equations (7) and (11) are satisfied by

\[
\lambda_{ij} = a \lambda \lambda_j + b \mu \mu_j + c \lambda \mu_i
\]  
(12)

where \( a, b, \) and \( c \) are constants and \( \mu_i \) is a vector such that

\[
\lambda^i \mu_i = 0.
\]  
(13)

When equation (12) is multiplied by \( g^{ik} \) and summed for \( i \) and it is noted that the covariant derivative of \( g^{ik} \) is equal to zero, the result is

\[
g^{ik} \lambda_{ij} = (g^{ik} \lambda_k)_j = \lambda^k_j = a \lambda^k \lambda_j + b \lambda^k \mu_j + c \lambda \mu^k.
\]

When in this equation \( k \) is replaced by \( i \), the result is

\[
\lambda^i_j = a \lambda^i \lambda_j + b \lambda^i \mu_j + c \lambda \mu^i.
\]  
(14)

We take analogously to equation (12), the equation

\[
\mu_{ij} = e \mu \mu_j + f \mu \lambda_j + g \lambda \mu_i
\]  
(15)

where \( e, f, \) and \( g \) are constants.

When equation (13) is differentiated covariantly with respect to \( x^i \) and in the resulting equation

\[
\lambda^i_{j} \mu_k + \lambda^i \mu_{i,j} = 0
\]

\( \lambda^i_j \) and \( \mu_{i,j} \) are replaced by their expressions from (14) and (15), by means of equations (10) and (13), the result is
\[ \mu_i \mu_i = 0. \]  

When this equation is differentiated covariantly with respect to \( x^i \) the result is to within the factor 2 the equation

\[ \mu_i \mu_i' = 0 \]

which is satisfied by the expression (15) for \( \mu_i \mu_j \) by means of equations (13) and (16).

4. When equation (12) is differentiated covariantly with respect to \( x^i \) and in the resulting equation

\[ \lambda_{i,jk} = a_2 \lambda_{i,k} + a_1 \lambda_{j,k} + b \mu_{i} \lambda_{j,k} + c \mu_{j} \lambda_{i,k} + b \lambda_{i,j} + c \lambda_{i} \mu_{j,k} \]

the covariant derivatives of the \( \lambda_i \)'s are replaced by expressions of the type (12), the result is

\[ \lambda_{i,jk} = 2a_2 \lambda_{i} \lambda_{j} \lambda_{k} + 2ab \lambda_{i} \lambda_{j} \mu_{k} + (ab + ac) \lambda_{i} \mu_{j} \lambda_{k} + b^2 \lambda_{i} \mu_{j} \mu_{k} \]

\[ + 2ac \mu_{i} \lambda_{j} \lambda_{k} + bc \mu_{i} \lambda_{j} \mu_{k} + (bc + c^2) \mu_{i} \mu_{j} \mu_{k} + \lambda_{i} \mu_{j} + c \lambda_{i} \mu_{j,k} \].

When this expression for \( \lambda_{i,jk} \) and the one obtained from it on interchanging \( j \) and \( k \) are substituted in the equation

\[ \lambda_{i,jk} - \lambda_{i,kj} = \lambda_{i} R_{ijk}^{h} \]

the result is

\[ \lambda_{i,R}^{h} = (ab - ac) \lambda_{i} \lambda_{j} \mu_{k} - (ab - ac) \lambda_{i} \mu_{j} \lambda_{k} + c \lambda_{i} \mu_{j} \mu_{k} \]

\[ + c^2 \mu_{i} \mu_{j} \lambda_{k} + b \lambda_{i} (\mu_{j,k} - \mu_{i,j}) + c \lambda_{i} \lambda_{j,k} - \lambda_{i} \mu_{j,k}. \]  

When the expressions for the covariant derivatives of \( \lambda_i \)'s of the type of equation (15) are substituted in this equation, the result is

\[ \lambda_{i,R}^{h} = (ab - ac + bg - bf + cg) \lambda_{i} (\lambda_{i} \mu_{k} - \lambda_{k} \mu_{i}) \]

\[ + (ce - e^2)\mu_{i} (\lambda_{i} \mu_{k} - \lambda_{k} \mu_{i}), \]

For \( f = g = 0 \) and \( ab - ac \) replaced by \( ab \) this equation becomes

\[ \lambda_{i,R}^{h} = ab \lambda_{i} (\lambda_{i} \mu_{k} - \lambda_{k} \mu_{i}) + (ce - e^2)\mu_{i} (\lambda_{i} \mu_{k} - \lambda_{k} \mu_{i}). \]  

Introduce vector \( \nu^{h} \) for which

\[ \nu^{h} \lambda_{h} = 1. \]

When \( \nu^{h} \lambda_{h} \) is inserted as a multiplier of the right-hand member of equation (19), the resulting equation is satisfied by

\[ R_{i,jk}^{h} = \nu^{h} [ab \lambda_{i} (\lambda_{i} \mu_{k} - \lambda_{k} \mu_{i}) + (ce - e^2)\mu_{i} (\lambda_{i} \mu_{k} - \lambda_{k} \mu_{i})]. \]

When equation (21) is contracted for \( h \) and \( k \), the result is

\[ R_{ij} = ab \nu^{h} \lambda_{i} \lambda_{j} - ab \lambda_{i} \mu_{j} + (ce - e^2)\mu_{i} (\lambda_{i} \nu^{h} \mu_{h} - \mu_{j}). \]

When this equation is multiplied by \( g^{ij} \) and summed for \( i \) and \( j \) in accordance with equations (10), (13), and (16) the result is

\[ R = g^{ij} R_{ij} = 0. \]

5. Einstein stated\(^2\) that the law of propagation of light according to general
relativity is characterized by the equation
\[ ds^2 = g_{ij}dx^idx^j = 0, \]
from which our equation (1) follows.

The Einstein equation of general relativity for light is
\[
R_{ij} - \frac{1}{2}g_{ij}R = -k\sigma g_{ij}\frac{d\chi^i}{ds}g_{jm}\frac{d\chi^m}{ds}
\]
where \( k \) is a constant connected with the Newtonian gravitation constant and \( \sigma \) is the density of ponderable matter at rest. This equation becomes by means of equation (3)
\[
R_{ij} - \frac{1}{2}g_{ij}R = -k\sigma\lambda_i\lambda_j.
\] (24)

When this equation is multiplied by \( g^{ij} \) and summed for \( i \) and \( j \), in accordance with the equation\(^8\)
\[ g^{ij}g_{ij} = 4 \]
and the fact that \( \lambda^i \) is a null-vector, the result is
\[ R = 0 \] (25)
and equation (24) becomes
\[ R_{ij} = -k\sigma\lambda_i\lambda_j. \] (26)

We replace equation (22) by the two equations
\[ R_{ij} = ab\nu^i\nu^j\lambda^i\lambda^j \] (27)
and
\[ ab\nu^i\nu^j + (e^2 - ce)e\nu^i\nu^j = 0. \] (28)

When \( i \) and \( j \) are interchanged in this equation, the result is
\[ ab\nu^i\nu^i + (e^2 - ce)e\nu^i\nu^i = 0 \]
which holds since equation (22) should be symmetric in \( i \) and \( j \). When this equation is subtracted from equation (28) the result is
\[ [ab - (e^2 - ce)e]\nu^i = 0. \]
Accordingly we take
\[ (e^2 - ce)e\nu^i = ab \] (29)
and equation (28) becomes
\[ ab\nu^i\nu^j + ab\nu^i\nu^j + \mu\nu^i = 0. \] (30)
When this equation is multiplied by \( \nu^i\nu^j \) and summed for \( i \) and \( j \), in accordance with equation (20) the result is
\[ ab\nu^i\nu^j + ab\nu^i\nu^j + \nu^i\nu^j\nu^i\nu^j = 0. \]
from which it follows that
\[ v^\mu \nu = -2ab \quad (31) \]
and equation (27) becomes
\[ R_{ij} = -2a^2b^2 \lambda i \lambda j. \quad (32) \]
This equation becomes the Einstein equation (26) for
\[ 2a^2 = k \quad (33) \]
which completes the solution.

3 R. G., equation (11.8), p. 28.
4 R. G., equation (11.3), p. 27.
8 Ibid., equations (96), p. 84, and (102), p. 88.

ON TAYLOR'S THEOREM IN SEVERAL VARIABLES*

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If an analytic system of partial differential equations in \( n \) independent variables \( x^1, \ldots, x^n \) is in involution in the sense of Cartan at the point \( P \) with coordinates \( (0, \ldots, 0) \), then for any dependent variable \( f \) the values
\[ f(P), \quad (\partial f/\partial x^1)(P), \quad (\partial^2 f/\partial x^i \partial x^j)(P), \ldots \]
are uniquely determined by appropriate Cauchy data, and one can apply the Cauchy-Kowalewski theorem to prove that the resulting power series in \( x^1, \ldots, x^n \) converges in some neighborhood of \( P \); this is essentially the Cartan-Kähler theorem. However, it frequently happens that the given system occurs naturally in terms of certain vector fields \( L_1, \ldots, L_n \) other than \( \partial/\partial x^1, \ldots, \partial/\partial x^n \); for example, \( L_1, \ldots, L_n \) may correspond to characteristics. In this case \( f(P), L_1 f(P), L_1 L_2 f(P), \ldots \) are also uniquely determined as before, and one would like to use these values directly to develop \( f \) in a power series without recourse to an \textit{ad hoc} coordinate system.

If the vector fields \( L_1, \ldots, L_n \) are linearly independent and commute with one another, then there exist unique analytically independent functions \( y^1, \ldots, y^n \) in a neighborhood of \( P \) such that \( y^1(P) = \ldots = y^n(P) = 0 \) and \( (dy^1, \ldots, dy^n) \) is the dual of \( (L_1, \ldots, L_n) \); in this case \( L_1 = \partial/\partial y^1, \ldots, L_n = \partial/\partial y^n \) and there is no problem. In general the dual of \( (L_1, \ldots, L_n) \) contains differential forms which