SYSTEMS OF DIFFERENTIAL EQUATIONS WITHOUT LINEAR TERMS*

BY COURTNEY COLEMAN

HARVEY MUDD COLLEGE

Communicated by S. Lefschetz, August 28, 1961

1. Consider the systems,

\[ \frac{dx}{dt} = f(x) + g(x,t) \]  
\[ \frac{dx}{dt} = f(x), \]

where:  
(a) \( x \) is an \( n \)-vector;  
(b) \( f(x) \) is homogeneous of positive degree \( k \);  
(c) \( f(x) \) and \( g(x,t) \) are continuously differentiable in \( x \) and in \( t \) for all \( x \) in a region \( D \) containing the origin and for all \( t \);  
(d) \( g(x,t) = o(\|x\|^k) \) in \( D \) uniformly in \( t \).  

If \( k = 1 \), then \( f(x) = Ax \) for some constant matrix \( A \).  
In this case, there are a number of well-known theorems\(^1\) whose hypotheses contain various assumptions about the characteristic roots of \( A \).  
In particular, there is the theorem of Lyapunov.

THEOREM (LYAPUNOV).  
If the characteristic roots of \( A \) have negative real parts,  
then the trivial solution, \( x = 0 \), of (1) is asymptotically stable.

In this note, we shall indicate how the hypotheses about the characteristic roots can be extended in a natural way to the case that \( k \geq 1 \).  
Lyapunov's theorem can then be generalized (see below).  
It seems likely that the generalized hypotheses used here can also be used in extending many of the classical theorems to the case where the system of first approximation (2) is no longer linear.

2. First, new variables are introduced.  
Let \( u = \|x\| \), where \( \|x\| \) represents the Euclidean norm of \( x \), and let \( y = u^{-1}x \).  
For \( k \geq 1 \), \( \|x\| > 0 \), system (2) becomes

\[ (a) \quad \frac{dy}{d\tau} = f(y) - (y,f(y))y, \]  
\[ (b) \quad \frac{du}{d\tau} = (y,f(y))u, \]

where \( (y,f(y)) \) is the scalar product of \( y \) and \( f(y) \) and \( d\tau = u^{k-1}dt \).  
System (3) is to be considered for \( u = 0 \) even though the change of variables is undefined for \( u = 0 \).  
Each solution, \( y = y(\tau,0,y^0) \), where \( y^0 = y(0,0,y^0) \), of (3a) determines a one-parameter family of solutions of (3b), \( u = u_0 \exp \int_0^\tau (y(t,0,y^0),f(y(t,0,y^0)))dt = u(\tau,0,y^0,u_0) \), where \( u_0 \) is the parameter.  
It is always assumed that \( u_0 \geq 0 \) and, hence, that \( u(\tau,0,y^0,u_0) \geq 0 \) for all \( \tau \).  
If \( u_0 \) and \( y^0 \) are given, then \( y = y(\tau,0,y^0) \), \( u = u(\tau,0,y^0,u_0) \) is the solution of (3) for which \( y^0 = y(0,0,y^0) \), \( u_0 = u(0,0,y^0,u_0) \).

Note that \( u = 0 \) defines an integral surface \( I \) in the \( y,u \)-space.

Definition 1:  
If \( y^0 \) belongs to the positive limit set of some solution of (3a), then \( y = y(\tau,0,y^0) \), \( u = u(\tau,0,y^0,u_0) \) is called a limit solution of (3).

Definition 2:  
The radial type number, \( \lambda(y^0) \), of a solution,

\[ y = y(\tau,0,y^0), \quad u = u(\tau,0,y^0,u_0) \quad (u_0 > 0) \]

of (3) is the number \( \lim_{\tau \to +\infty} r^{-1} \ln u(\tau,0,y^0,u_0) \).

Note that \( \lambda(y^0) \) is independent of \( u_0,u_0 > 0 \), that \( \lambda(y^0) \) may be infinite, and that \( \lambda(y^0) = \lambda(y^1) \) if \( y^1 = y(\tau_1,0,y^0) \) for some \( \tau_1 \).  
An equivalent definition of \( \lambda(y^0) \) is as follows.
Definition 3: \( \lambda(y^0) = -\sup \{ \lambda': u(\tau,0,u_0,y^0) \exp \lambda' \tau \text{ is bounded for all } \tau, 0 \leq \tau < +\infty \} \).

Type numbers (or their negatives, Lyapunov numbers) are usually defined somewhat more generally than this, but only the radial type numbers defined above are needed here. Now, any hypothesis concerning the real parts of the characteristic roots of \( A \) when \( k = 1 \) can be formulated in terms of the radial type numbers of the limit solutions when \( k \geq 1 \). For it can be shown that when \( k = 1 \), the set of radial type numbers of the limit solutions of (3) is precisely the set of real parts of the characteristic roots of \( A \). In particular, Lyapunov's theorem can be generalized.

Theorem. If \( k \geq 1 \) and if the radial type number of every limit solution of (3) is negative, then the trivial solution, \( x = 0 \), of (1) is asymptotically stable.

The proof consists of showing that the hypothesis implies that the radial type number of every solution of (3) is negative and that this in turn implies that the trivial solution of (2) is asymptotically stable. A theorem of Zubov is used to prove this latter assertion. Finally, it is a theorem of Massera that if the trivial solution of (2) is asymptotically stable, so is the trivial solution of (1).

3. The results of an earlier paper and some remarks made by S. Lefschetz were the stimuli for what has been presented here. It should be noted that the main theorem of reference (5) is a special case of the theorem stated above.

* This research was supported in part by the National Science Foundation under contract number NSF G 12932.


---

ON 2-SPHERES IN 4-MANIFOLDS

BY MICHEL A. KERVAIRE AND JOHN W. MILNOR

INSTITUTE OF MATHEMATICAL SCIENCES, NEW YORK UNIVERSITY, AND DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY

Communicated by N. E. Steenrod, August 23, 1961

Let \( M^{2n} \) be a simply connected differentiable manifold, and let \( \xi \in \pi_n(M^{2n}) \) be a given homotopy class of maps \( S^n \to M^{2n} \). It is known that if \( n > 2 \), the class \( \xi \) can be represented by a differentiable imbedding \( f: S^n \to M^{2n} \). This follows from a reasoning similar to the one used by H. Whitney to prove that every differentiable \( n \)-manifold can be differentiably imbedded in Euclidean \( 2n \)-space. (Compare Milnor, Lemma 6.) It is also included in a more general theorem of A. Haefliger. Both arguments, however, break down for \( n = 2 \). This leads to the following question:

Let \( M^4 \) be a simply connected differentiable manifold. Is every element of \( \pi_4(M^4) \) representable by a differentiably imbedded sphere?