\[ p_1[M] = 3\sigma(M). \]

(Cf. Hirzebruch, ref. 4, Theorem 8.2.2, p. 85.) The proof of Lemma 1 is based on the fact that \( \pi_{n+4}(S^n) \) is cyclic of order 24 for \( n \geq 5. \)

\[ \dagger \] In the sense of J. H. C. Whitehead.\textsuperscript{10} Any two regular neighborhoods of \( K \) in \( M \) are combinatorially equivalent by reference 10, Theorem 23.


\[ 4 \] Hirzebruch, F., \textit{Neue Topologische Methoden in der Algebraischen Geometrie} (Berlin: Springer Verlag, 1956).


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**AN ALGORITHM FOR EQUILIBRIUM POINTS IN BIMATRIX GAMES**

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1. Recently N. N. Vorobjev\textsuperscript{1} has presented a constructive procedure for computing all equilibrium points for the case of bimatrix (i.e., finite two-person non-cooperative non-zero-sum) games. The purpose of the present note is to simplify his algorithm both in theory and application. In the terms of his paper, the classification of extreme equilibrium strategies into two types is eliminated, and the enumeration of all such strategies is reduced to a single routine.

2. For the sake of easy comparison with Vorobjev’s work, his notation will be used. If \( M \) is any matrix, \( M_i \) denotes the \( i \)th row of \( M \), \( M_j \) denotes the \( j \)th column of \( M \), and \( M^T \) denotes the transpose of \( M \). Furthermore, \( J_p \) denotes the \( p \)-dimensional vector with all components equal to one, and \( O_p \) denotes the \( p \)-dimensional vector with all components equal to zero. Inequalities between vectors are to hold in all components.

A \textit{bimatrix game} \( \Gamma \) is defined by two real \( m \) by \( n \) payoff matrices, \( A = (a_{ij}) \) and \( B = (b_{ij}) \): if player 1 chooses \( i \in \{1, \ldots, m\} \) and player 2 chooses \( j \in \{1, \ldots, n\} \), then player 1 is paid \( a_{ij} \) and 2 is paid \( b_{ij} \). Mixed strategies for 1 and 2 are probability vectors of dimension \( m \) and \( n \) and are denoted by \( X \) and \( Y \) respectively. Thus,

\[ XJ_m^T = 1, \ X \geq O_m \quad \text{and} \quad J_nY^T = 1, \ Y \geq O_n. \]
If 1 uses mixed strategy $X$ and 2 uses $Y$, then their expected payoffs are $XAY^T$ and $XBY^T$, respectively. An equilibrium point is a pair of mixed strategies $(\bar{X}, \bar{Y})$ such that

$$\bar{X}A \bar{P}^T \geq XA \bar{P}^T \quad \text{and} \quad \bar{X}B \bar{P}^T \geq XB \bar{P}^T$$

for all $X$ and $Y$. The set of all equilibrium points for $\Gamma$, which is nonempty by the theorem of Nash, will be denoted by $S_\Gamma$.

Clearly $(\bar{X}, \bar{Y}) \in S_\Gamma$ if and only if

$$\bar{X}A \bar{P}^T \geq A_i \bar{P}^T \quad \text{and} \quad \bar{X}B \bar{P}^T \geq \bar{X}B_j$$

for all $i$ and $j$.

3. Given any $X$, set $S(X) = \{Y \mid (X, Y) \in S_\Gamma\}$. Since $S(X)$ is the solution set of the finite system of linear inequalities,

$$XAY^T \geq A_i Y^T \quad (i = 1, \ldots, m)$$

$$XBY^T \geq XB_j \quad (j = 1, \ldots, n)$$

$$Y \geq O, \quad J_n Y^T = 1,$$

it is clear that $S(X)$ is a compact convex set, which is possibly empty (compare the Lemma, pp. 319–320 of ref. 1). For any set $\mathcal{X}$ of mixed strategies $X$, set

$$S(\mathcal{X}) = \bigcap_{X \in \mathcal{X}} S(X).$$

Clearly, $S(\mathcal{X})$ is also a compact convex set, which is possibly empty. For any set $\mathcal{S}$, $K(\mathcal{S})$ will denote the set of extreme points of $\mathcal{S}$.

**Definition.** The mixed strategy $Y$ is called an extreme equilibrium strategy if

$$Y \in K\left( \bigcap_{k=1}^p S(X_k) \right) = K(S(\mathcal{X}))$$

for some finite set $\mathcal{X} = \{X_1, \ldots, X_p\}$ of mixed strategies.

It should be clear that analogous definitions can be given with the roles of the players reversed.

4. The results of Vorobjev can now be stated as follows:

**Theorem 1.** It is possible to enumerate effectively finite sets $\mathcal{X}$ and $\mathcal{Y}$, which contain all extreme equilibrium strategies for the two players.

**Theorem 2.** For any finite set $\mathcal{X}$, it is possible to describe effectively the set $S(\mathcal{X})$ by computing its extreme points, which are finite in number.

**Theorem 3.**

$$S_\Gamma = \bigcup_{\mathcal{X} \in \mathcal{X}} [\mathcal{X}] \times S(\mathcal{X}),$$

where $[\mathcal{X}]$ denotes the convex hull of $\mathcal{X}$.

Vorobjev has derived Theorems 2 and 3 from Theorem 1 in a very elegant manner. His proofs will be repeated here to make this account self-contained.

**Proof of Theorem 2:** Since $S(\mathcal{X})$ is a compact convex set, to describe $S(\mathcal{X})$ we need only find $K(S(\mathcal{X}))$. However, $K(S(\mathcal{X})) \subset \mathcal{Y}$, since $\mathcal{Y}$ contains all extreme equilibrium strategies. Set $\mathcal{Y} = S(\mathcal{X}) \cap \mathcal{Y}$. To verify whether a point $Y$ of $\mathcal{Y}$ also lies in $S(\mathcal{X})$, we need only test the finite set of inequalities $XAY^T \geq A_i Y^T, XBY^T \geq$
for the finite set of points \( X \in \mathcal{X} \). Since this process is finite, \( \mathcal{Y} \) is effectively enumerable.

Since \( K(\mathcal{S}(\mathcal{X})) \subseteq \mathcal{Y} \subseteq \mathcal{S}(\mathcal{X}) \), upon deleting those points of \( \mathcal{Y} \) which are convex combinations of other points of \( \mathcal{Y} \), the set \( K(\mathcal{S}(\mathcal{X})) \) remains. Since \( \mathcal{Y} \) is a finite set, this process of deletion is effective.

**Proof of Theorem 3**: Let \((X, Y) \in \mathcal{S}_r\). Then \( K(\mathcal{S}(Y)) \subseteq \mathcal{X} \), because \( \mathcal{X} \) contains all extreme equilibrium strategies. Let \( \mathcal{X} \) denote the finite set \( K(\mathcal{S}(Y)) \). If \( X \in \mathcal{S}(Y) \), then \( X \in [K(\mathcal{S}(Y))] = [\mathcal{X}] \) because \( \mathcal{S}(Y) \) is a compact convex set. Clearly \( \mathcal{X} \subset \mathcal{S}(Y) \) and hence \( Y \in \mathcal{S}(X^*) \) for all \( X^* \in \mathcal{X} \). This means \( Y \in \mathcal{S}(\mathcal{X}) = \bigcap_{X^* \in \mathcal{X}} \mathcal{S}(X^*) \).

Conversely, let \( X \in [\mathcal{X}] \) and \( Y \in \mathcal{S}(\mathcal{X}) \) for some finite set \( \mathcal{X} \subset \mathcal{X} \). Then \( \mathcal{X} \subset \mathcal{S}(Y) \) and \( \mathcal{X} \subset \mathcal{S}(Y) \) because \( \mathcal{S}(Y) \) is convex. Therefore, \( X \in \mathcal{S}(Y) \), which means \((X, Y) \in \mathcal{S}_r\).

5. **Proof of Theorem 1**: The proof will be based on the following two lemmas.

**Lemma 1.** Let \( \bar{Y} \) be an extreme equilibrium strategy. Set \( \bar{\alpha} = \max_i A_i \bar{Y}^T \).

Then \((\bar{Y}, \bar{\alpha})\) is an extreme solution of the following system:

\[
\alpha J^T_m \geq A Y^T, \quad Y \geq O_n, \quad J_n Y^T = 1.
\]

**Proof:** For \( \bar{Y} \) to be an extreme equilibrium strategy means that there exists a finite set \( \mathcal{X} = \{X_1, \ldots, X_p\} \) such that \( \bar{Y} \) is an extreme solution of

\[
X_k A Y^T \geq A_i Y^T \quad (i = 1, \ldots, m; \quad k = 1, \ldots, p)
\]

\[
X_k B Y^T \geq X_k B_j \quad (j = 1, \ldots, n; \quad k = 1, \ldots, p)
\]

\[
Y \geq O_n, \quad J_n Y^T = 1.
\]

Clearly, \( X_k A \bar{Y}^T = \bar{\alpha} \) for \( k = 1, \ldots, p \).

Suppose \( \bar{Y} = \frac{1}{2}(Y' + Y^*) \) and \( \bar{\alpha} = \frac{1}{2}(\alpha' + \alpha^*) \), where \((Y', \alpha')\) and \((Y^*, \alpha^*)\) are distinct solutions to the system

\[
\alpha \geq A_i Y^T \quad (i = 1, \ldots, m)
\]

\[
X_k B Y^T \geq X_k B_j \quad (j = 1, \ldots, n; \quad k = 1, \ldots, p)
\]

\[
Y \geq O_n, \quad J_n Y^T = 1.
\]

Then,

\[
\bar{\alpha} = \max_i A_i \bar{Y}^T \leq \frac{1}{2}(\max_i A_i Y'^T + \max_i A_i Y^*T) \leq \frac{1}{2}(\alpha' + \alpha^*) = \bar{\alpha},
\]

and hence,

\[
\max_i A_i Y'^T = \alpha', \quad \max_i A_i Y^*T = \alpha^*.
\]

On the other hand,

\[
\bar{\alpha} = X_k A \bar{Y}^T = \frac{1}{2}(X_k A Y'^T + X_k A Y^*T) \leq \frac{1}{2}(\alpha' + \alpha^*) = \bar{\alpha},
\]

and hence,

\[
X_k A Y'^T = \alpha', \quad X_k A Y^*T = \alpha^*
\]

for \( k = 1, \ldots, p \). However, this proves \( Y' \in \mathcal{S}(\mathcal{X}) \) and \( Y^* \in \mathcal{S}(\mathcal{X}) \). Since \( \bar{Y} = \)}
\( \frac{1}{2}(Y' + Y'') \) is extreme in \( S(\mathcal{X}) \), this implies \( Y' = Y'' = \bar{Y} \) and hence \( \alpha' = \alpha'' = \bar{\alpha} \). However, this contradicts \( (Y', \alpha') \neq (Y'', \alpha'') \).

Therefore, \( \bar{Y} \) is an extreme solution to the system in which the inequalities

\[
X_k B Y^T \geq X_k B \cdot \ (j = 1, \ldots, n; \ k = 1, \ldots, p)
\]

are adjoined to the system of the lemma. However, these merely require that \( y_j = 0 \) for those \( j \) for which \( X_k B \cdot \leq \max X_k B \cdot \) for some \( k \). Call this set \( \mathcal{N} \).

Suppose \( \bar{Y} = \frac{1}{2}(Y' + Y'') \), where \( Y' \) and \( Y'' \) are distinct solutions to the system of the lemma. Since \( Y' \geq \bar{O}_n, Y'' \geq \bar{O}_n \), and \( \bar{y}_j = 0 \) for \( j \in \mathcal{N} \), we have \( y_j = 0 \) for \( j \in \mathcal{N} \) and hence \( Y' \) and \( Y'' \) also solve the enlarged system. This contradiction proves the lemma.

**Lemma 2.** Let \((\bar{Y}, \bar{\alpha})\) be an extreme solution of the system

\[
\alpha J_m^T \geq A Y^T, \ Y \geq \bar{O}_n, \ J_n Y^T = 1.
\]

Then there exists an \( s \) by \( s \) submatrix \( D \) of \( A \) such that

\[
D = \begin{pmatrix} D & -J_s^T \\ J_s & 0 \end{pmatrix}
\]

is nonsingular. Furthermore, renumbering rows and columns, if necessary, to place \( D \) in the upper left corner of \( A \),

\[
\begin{align*}
\bar{y}_j &= \frac{\sum_{i=1}^s D_{ij}}{|D|} \quad (j = 1, \ldots, s) \\
\bar{y}_j &= 0 \quad (j = s + 1, \ldots, n) \\
\bar{\alpha} &= \frac{|D|}{|D|} = \frac{\sum_{i,j=1}^s D_{ij}}{|D|} \\
(D_{ij} \text{ denotes the cofactor of } a_{ij} \text{ in } D; \ |D| \text{ and } |D| \text{ denote the determinants of } D \text{ and } D).
\end{align*}
\]

**Proof:** If \((\bar{Y}, \bar{\alpha})\) is an extreme solution to

\[
\alpha J_m^T \geq A Y^T, \ Y \geq \bar{O}_n, \ J_n Y^T = 1,
\]

then \( A_i \bar{Y}^T = \bar{\alpha} \) for some \( i \). Reindex rows, if necessary, so that \( A_i \bar{Y}^T = \bar{\alpha} \) for \( i = 1, \ldots, r \) and \( A_i \bar{Y}^T < \bar{\alpha} \) for \( i = r + 1, \ldots, m \). Since \( \bar{Y} \geq \bar{O}_n \) and \( J_n \bar{Y}^T = 1 \), \( \bar{y}_j > 0 \) for some \( j \). Reindex columns, if necessary, so that \( \bar{y}_j > 0 \) for \( j = 1, \ldots, s \) and \( \bar{y}_j = 0 \) for \( j = s + 1, \ldots, n \).

I contend that the system of equations

\[
\begin{align*}
\sum_{j=1}^s a_{ij} y_j &= \alpha \quad (i = 1, \ldots, r) \\
\sum_{j=1}^s y_j &= 1
\end{align*}
\]

has the unique solution \( y_j = \bar{y}_j \) for \( j = 1, \ldots, s \) and \( \alpha = \bar{\alpha} \). Suppose, to the contrary, that there are distinct solutions \((y_1', \ldots, y_s', \alpha')\) and \((y_1'', \ldots, y_s'', \alpha'')\) and define \( \bar{Y}' \) and \( \bar{Y}'' \) by...
where $\epsilon > 0$ is to be chosen. Since $\bar{y}_j > 0$ for $j = 1, \ldots, s$ it is clear that $\bar{y}' \geq 0$ and $\bar{y}^s \geq 0$ for $\epsilon$ sufficiently small. Furthermore, since $\sum_{j=1}^s \bar{y}_j = \sum_{j=1}^n \bar{y}_j = 1$, we have $J_n \bar{y}'^T = 1$ and $J_n \bar{y}^s^T = 1$.

For these mixed strategies, we have

$$A_i \bar{y}'^T = \bar{\alpha} + \epsilon(\bar{\alpha}' - \alpha^s), \quad A_i \bar{y}^s^T = \bar{\alpha} + \epsilon(\alpha^s - \bar{\alpha}') \quad (i = 1, \ldots, r)$$

and

$$A_i \bar{y}'^T = A_i \bar{y}^T + \epsilon \sum_{j=1}^s a_{ij}(\bar{y}_j - y_j^s), \quad A_i \bar{y}^s^T = A_i \bar{y}^T + \epsilon \sum_{j=1}^s a_{ij}(y_j^s - \bar{y}_j) \quad (i = r + 1, \ldots, m).$$

Since $A_i \bar{y}' < \bar{\alpha}$ for $i = r + 1, \ldots, m$, it is possible to choose a fixed $\delta$ with the same sign as $\bar{\alpha}' - \alpha^s$ such that

$$(\bar{\alpha} + \delta)J_m^T \geq A \bar{y}'^T, \quad (\bar{\alpha} - \delta)J_m^T \geq A \bar{y}^s^T$$

for some $\epsilon > 0$ sufficiently small. Hence, $(\bar{y}', \bar{\alpha} + \delta)$ and $(\bar{y}^s, \bar{\alpha} - \delta)$ solve the system of the lemma. However, $(\bar{y}, \bar{\alpha}) = \frac{1}{2}[(\bar{y}', \bar{\alpha} + \delta) + (\bar{y}^s, \bar{\alpha} - \delta)]$ is an extreme solution. Hence, $\delta = 0$, which implies $\alpha' = \alpha^s$, and $\bar{y}' = \bar{y}^s$, which implies $(y'_1, \ldots, y'_r) = (y^s_1, \ldots, y^s_s)$. This proves the contention made above.

The remainder of the proof is standard.\textsuperscript{3} The uniqueness of the solution to the equation system implies that the columns of the matrix

$$\begin{pmatrix} a_{11} & \ldots & a_{1s} & -1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{r1} & \ldots & a_{rs} & -1 \\ 1 & \ldots & 1 & 0 \end{pmatrix}$$

are linearly independent. Hence we can choose $s$ rows (including the last row) so that

$$D = \begin{pmatrix} a_{11} & \ldots & a_{1s} & -1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{r1} & \ldots & a_{rs} & -1 \\ 1 & \ldots & 1 & 0 \end{pmatrix}$$

is nonsingular (the rows may have to be renumbered again). We shall call the matrix

$$D = \begin{pmatrix} a_{11} & \ldots & a_{1s} \\ \vdots & \vdots & \vdots \\ a_{r1} & \ldots & a_{rs} \end{pmatrix}$$

which is a square submatrix of $A$, a kernel for the extreme solution $\bar{y}$ (not "the" kernel since the steps in its construction are not unique). The formulas for $\bar{y}$ and $\bar{\alpha}$ then follow by Cramer's rule.
Since there are only a finite number of square submatrices of $A$, the proof of Theorem 1 follows by combining Lemmas 1 and 2. Note that not every kernel which provides an extreme solution to the system of Lemma 2 need provide an extreme equilibrium strategy. However, the finite set $\bar{q}$ consisting of all mixed strategies computed from the kernels certainly contains all extreme equilibrium strategies.


ORBIT SPACES OF FINITE ABELIAN TRANSFORMATION GROUPS*

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1.—We shall consider transformation groups—or actions—$(\pi, X)$ where $X$ is a locally compact Hausdorff space and $\pi = \beta_1 X \times \ldots \times \beta_r$, each $\beta_i$ being cyclic of prime order $p$. An action $(\pi, X)$ is free if the fixedpoint set $F(\gamma)$ is empty for each cyclic subgroup $\gamma$. The free part $\bar{g}$ of an action $(\pi, X)$ is the open set $X = \cup F(\gamma)$; the induced action $(\pi, \bar{g})$ is free. We shall consider the problem of determining the cohomology of the orbit spaces of free actions, in particular, the orbit spaces $\bar{g}/\pi$. In the case of cohomology spheres, we simplify and complete the computation given in an earlier note.

2. Some Cohomology Groups. An action $(\pi, X)$ being given, let $A'(X)$ be the $Z_p(\pi)$-module of Alexander-Spanier cochains on $X$, values in $Z_p$, modulo those of empty support, and let $A(X)$ be the submodule of compactly supported elements. $A(X)$ is identical with the module of compactly supported sections of the Alexander-Spanier sheaf on $X$, values in $Z_p$. Let $H(X) = H(A(X))$ where $H$ stands for cohomology, and for $\xi \in Z_p(\pi)$, let $H_\xi(X) = H(\xi A(X))$. Multiplication by an element $\eta$ in $Z_p(\pi)$ induces $\eta^*: H_\xi(X) \rightarrow H_{\xi \eta}(X)$. We have also $\iota^*: H_\xi(X) \rightarrow H_\xi(X)$ induced by the inclusion $\iota: A(X) \rightarrow A(X)$.

For $g \in \pi$, $g \neq 1$, let $\sigma(g) = 1 + g + \ldots + g^{p-1}$, $\tau(g) = 1 - g$ and let $\rho(g)$ be either one of $\sigma(g)$, $\tau(g)$ and $\bar{p}(g)$ the other. Evidently, $\rho(g)\bar{p}(g) = 0$.

Let $\gamma$ be the cyclic subgroup generated by $g$. The support of each element of $\rho(g)A(X)$ lies in $X - F(\gamma)$. In fact, if $x \in F(\gamma)$, $c \in A(X)$, we have $(gc)(x) = c(g(x)) = c(x)$. Hence, $(\sigma c)(x) = p c(x) = 0$, $(\tau c)(x) = c(x) - c(x) = 0$; so $\rho c$ ($\rho = \rho(g)$) vanishes on $F(\gamma)$. If $\gamma$ acts trivially on $X$, then $\rho^*$ annihilates $H(X)$, since in this case $F(\gamma) = X$.

Let $\xi_0 = X$ and for $s = 1, \ldots, r$ let $\xi_s = X - \cup F(\beta_i)$. We have inclusions $A(\xi_s) \subset A(X)$, $\rho_1 \ldots \rho_h A(\xi_s) \rightarrow \rho_1 \ldots \rho_h A(\xi)$. The second of these is bijective. For let $c \in A(X)$. The support of $p_1 \ldots p_\xi c$ is in $X - F(\beta_i), i = 1, \ldots, s$, hence is in $\xi_s$. Therefore, $\rho_1 \ldots \rho_\xi c$ can be identified with an element of $\rho_1 \ldots \rho_s A(\xi_s)$. It is