This work was supported by a grant from the National Science Foundation.

Part of the computation work was carried out at the M. I. T. Computation Center.

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ON BLOCKS OF REPRESENTATIONS OF FINITE GROUPS

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Communicated October 20, 1961

The blocks of representations of a finite group $G$ have been studied in several
previous papers. Here, a number of further results are given which are needed for
application of this theory to an investigation of the structure of groups of even
order.

1. Let $G$ be a group of finite order $g$ and let $p$ be a fixed prime number. The
irreducible characters $\chi_1, \chi_2, \ldots, \chi_k$ are distributed into disjoint sets, the $p$-blocks
of $G$. We shall denote by $G^p$ the set of $p$-regular elements of $G$ and by $\chi^p_\mu$ the
restriction of $\chi_\mu$ to $G^p$. If $B$ is a fixed block, the functions $\chi^p_\mu$ with $\chi_\mu \in B$
generate a module $M_B$ with regard to the ring $\mathbb{Z}$ of integers. By a basic set $\varphi_B$ for $B$, we mean
any basis $\{ \varphi_\mu \}$ of $M_B$. Thus,

$$\chi^p_\mu = \sum \phi_\mu \varphi_\mu \quad \text{for} \quad \chi_\mu \in B$$

with $d_{\mu \rho} \in \mathbb{Z}$. For $\varphi_\mu, \varphi_\rho \in \varphi_B$, set

$$c_{\mu \rho} = \sum \phi_\mu \phi_\rho$$

the sum extending over all $\mu$ with $\chi_\mu \in B$. Hence $(c_{\mu \rho})$ is the matrix of a quadratic
form $Q$. If $\varphi_B$ is replaced by another basic set, $Q$ is replaced by an equivalent
quadratic form.

It is well known that the irreducible modular characters in $B$ form a basic set.
In this case, the $d_{\mu \rho}$ are the decomposition numbers and the $c_{\mu \rho}$ are the Cartan in-
variants of $B$. We shall use the same terms in the case of an arbitrary basic set.

Let $d$ be the defect of $B$. We consider $p^d$ as fixed and shall say that a quantity is
bounded if it is bounded for all $p$-blocks of defect $d$ of all finite groups with bounds
depending only on $p^d$. In this sense, the dimension $r$ of $M_B$ and $\det(c_{\mu})$ are bounded.
Hence, $Q$ belongs to one of a finite number of classes of positive definite quadratic forms. As a consequence, we have

**Theorem 1.** There exist bounds $\gamma = \gamma(p^d)$ depending only on $p^d$ such that for each $p$-block $B$ of defect $d$ of a finite group, a basic set can be chosen such that the Cartan invariants are at most equal to $\gamma$.

We remark that the union of basic sets $\varphi_B$ for all $p$-blocks $B$ of $G$ is still linearly independent.

2. Let $P$ be a $p$-element of $G$, i.e., an element whose order $p^s$ is a power of $p$. By the *section* $S_P$ of $P$, we mean the set of elements of $G$ conjugate to an element $PR$, where $R$ is a $p$-regular element of the centralizer $C(P)$ of $P$ in $G$: $R \in C(P)^o$. Thus, if $P$ ranges over a set of representatives for the conjugate classes of $p$-elements in $G$, each element of $G$ belongs to exactly one $S_P$. If $R \in C(P)^o$, we can set

$$x_\mu (PR) = \sum_b \sum_\rho d_{\mu \rho}^P \varphi_\rho^P (R).$$

Here, $b$ ranges over the $p$-blocks of $C(P)$ and, for each $b$, $\rho$ ranges over the indices of the elements $\varphi_\rho^P$ of a basic set $\varphi_b$. Moreover, the $d_{\mu \rho}^P$ are algebraic integers of the field of $p^s$th roots of unity which do not depend on $R$. We call the $d_{\mu \rho}^P$ the generalized decomposition numbers.

Each $p$-block $b$ of $C(P)$ determines a $p$-block $B = bG$, cf. the papers quoted in reference 1. If we consider only $x_\mu \in B$ in (3), it suffices to let $b$ range over those blocks of $C(P)$ for which $bG = B$. For $c_{\mu \rho}^P$, $\varphi_\rho^P \in \varphi_b$,

$$\sum_\mu d_{\mu \rho}^P c_{\mu \rho}^P = c_{\rho \rho}^P,$$

where $\mu$ ranges over the indices for which $x_\mu \in B$ and where $c_{\mu \rho}^P$ is the Cartan invariant of $b$. On the other hand, for the same range of $\mu$,

$$\sum_\mu d_{\mu \rho}^P d_{\mu \rho'}^P = 0$$

when $P$ and $P'$ are two nonconjugate $p$-elements or when $P = P'$ but $\varphi_{\mu}^P$, $\varphi_{\nu}^P$ belong to basic sets of two distinct $p$-blocks of $C(P)$.

If $bG = B$, the defect $d_b$ of $b$ is at most equal to the defect $d$ of $B$. If the basic set $\varphi_b$ is chosen in accordance with Theorem 1, $|c_{\mu \rho}^P| \leq \gamma(p^d)$. It follows from (4) that we have only finitely many possibilities for the matrix $(d_{\mu \rho}^P)$. This leads to

**Theorem 2.** For a given $p^d$, there exist a finite number of possible types of $p$-blocks $B$ of defect $d$. For each type, the set of generalized decomposition numbers $\{d_{\mu \rho}^P\}$ is completely determined assuming that suitable basic sets are used. If the group $C(P)$ is given, the values of the $x_\mu \in B$ for elements of the section of $P$ are determined.

We may assume that, for a fixed type, the defect group $D$ of the block is given as an abstract $p$-group and that it is also given which conjugate classes of $D$ are "fused" in $G$, i.e., are included in the same conjugate class of $G$. If $P$ is not conjugate to an element of $D$, then each $x_\mu \in B$ vanishes on $S_P$. In the last part of Theorem 2, it is assumed that we know to which element of $D$ (if any) $P$ is conjugate.

3. Estimates for the number of types for given $p^d$ obtainable by the previous method would be extremely large. There are other methods available which also give Theorems 1 and 2 and which yield better results. These are based on the follow-
ing remarks: 1. The discussion of the Cartan invariants of $C(P)$ can be reduced to the same discussion for $C(P)/\{P\}$. If $P \neq 1$, the defect is reduced. We can therefore use an inductive procedure to obtain the possibilities for the $d_{\rho\mu}$ with a fixed $P \neq 1$. 2. Consider the $d_{\rho\mu}$ with fixed $\rho$ and $P$ as the coefficients of a column $b_\mu^P$ with $\mu$ as row index. The columns with coefficients in $\mathbb{Z}$ which are orthogonal to all columns $b_\mu^P$ with $P \neq 1$ form a $\mathbb{Z}$-module $X$. By (5), $b_\mu^1 \subseteq X$. Any $\mathbb{Z}$-basis of $X$ can be used for the set of columns $b_\mu^1$, assuming a suitable choice of the basic set. 3. In addition to (4) and (5), there are a number of other results which facilitate the discussion. In particular, congruences for the columns $b_\mu^P$ can be established. Also, Theorems 4 and 5 of the third paper quoted in reference 1 can be used; these results can be refined further.

4. One of the $p$-blocks of $G$ must contain the principal character $\chi_0 = 1$. We term this block the principal block $B_0$ of $G$ and state a number of results for $B_0$.

**Theorem 3.** Let $b$ be a block of a subgroup $H$ of $G$, let $T$ be its defect group in $H$ and assume that the centralizer of $T$ in $G$ is included in $H$ so that $b^G$ is defined as a block of $G$. Then $b^G$ is the principal block $B_0$ of $G$ if and only if $b$ is the principal block of $H$.

It follows from this that for $B = B_0$, the sum $\sum_b$ in (3) consists only of the term $b = b_0$. This simplification is of importance for the applications.

By the $p$-regular core $K_p(G)$ of a group $G$, we mean the unique maximal normal subgroup of $G$ of an order prime to $p$.

**Theorem 4.** The intersection of the kernels of the irreducible representations in the principal $p$-block $B_0$ of $G$ is the $p$-regular core $K_p(G)$ of $G$.

This means that the principal $p$-block of $G$ can be identified with that of $G/K_p(G)$.

**Corollary.** If $n$ is the sum of the degrees of the irreducible characters $\chi_\rho \subseteq B_0$, then

$$n \leq (G:K_p(G)) \leq [2n]!.$$  

(6)

Indeed, the algebraic conjugates of $\chi_\rho \subseteq B_0$ lie again in $B_0$ and hence $\sum \chi_\rho \subseteq B_0$ is a rational character belonging to a faithful representation of $G/K_p(G)$. Now the right-hand part of (6) is obtained from a theorem of I. Schur\(^1\) (even in a somewhat sharper form). The other part is trivial.

**Theorem 5.** If $B$ is the principal block, the basic set $\varphi_B$ can be chosen such that the constant 1 appears in it and that all Cartan invariants $c_{\rho\mu}$ belonging to it lie below a bound $\gamma_0(p^n)$ where $p^n$ is the highest power of $p$ dividing $g$.

\(^*\) This research was supported by the United States Air Force under Contract No. AF 49 (638)-287 monitored by the Air Force Office of Scientific Research of the Air Research and Development Command.


2 Using results from the theory of quadratic forms, explicit estimates for $\gamma(p^d)$ can be given, but they are probably much too large.

3 Schur, I., Sitzungsberichte der Preussischen Akademie Berlin, Mathematisch-Naturwissenschaftliche Klasse, 77–91 (1905).