needed for the high-voltage ions, not provided for by the magnetic field in the schemes already being used so far without success.

Summary.—The Ionic Centrifuge is briefly described. The discharge in this device violates the usual rules of magnetohydrodynamics because of the high electric field parallel to the magnetic field in the space charge affected boundaries.\(^6,\)\(^8\) The kinetic energy of random motion of the particles is proportional to the voltage which the main discharge holds at each point. By causing this voltage to rise to a high positive value and then drop to zero at the cylinder, the ions are not collected when their kinetic energy is high, but only at the cylinder where this kinetic energy is low again. The suitability of this arrangement for nuclear power converters is pointed out.

\(^7\) Slepian, J., Physics of Fluids, 1, No. 6, 547 (1958).
\(^9\) Loeb, Leonard B., in Fundamental Processes of Electrical Discharges in Gas (New York: John Wiley and Sons, 1939), shows a reversal of electric gradient just when saturation limitation by space charge begins. Fig. 151, p. 317.

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**PRESSURE SHOCKS IN VISCOUS HEAT-CONDUCTING GASES**

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1. Introduction.—Let \( \Sigma(t) \) be a wave surface or boundary of a disturbance which is propagated in a viscous and heat-conducting gas. Since the gas is viscous, it is commonly assumed that the velocity is continuous over the surface \( \Sigma(t) \); this implies that the density is also continuous over \( \Sigma(t) \) on account of the following dynamical or shock condition (3). Thus, it is natural to assume that the pressure is likewise continuous over the surface \( \Sigma(t) \). But it can then be shown from the basic equations for the behavior of the gas (see section 3) that all first and higher-order derivatives of the velocity, density, and pressure must be continuous over the surface \( \Sigma(t) \); in other words, the surface \( \Sigma(t) \) fails to bear a discontinuity of any order and hence one is led to consider a wave of finite thickness. There is, however, no mathematical or physical necessity to require the continuity of the pressure over the surface \( \Sigma(t) \). Assuming a discrete discontinuity in the pressure while imposing the recognized condition that the velocity is continuous over the surface \( \Sigma(t) \), we arrive at the concept of the pressure shock which would appear to merit consideration when dealing with waves, e.g., blast waves, where pressure effects are of primary concern.

In this communication, we shall treat the problem of the propagation of pressure
shocks into an ideal gas which is uniform and at rest with viscosity and thermal conductivity different from zero. We assume the following boundary conditions, appropriate to this problem, namely,

\[ \dot{v}_i = 0; \quad \frac{d\bar{p}}{dn} = 0; \quad \frac{dT}{dt} = 0, \quad (1) \]

where the \( v_i \) are the velocity components, \( p \) is the pressure, and \( T \) is the temperature; the bar appearing in the relations (1) denotes evaluation on the rear or flow side of the shock surface \( \Sigma(t) \). The first set of relations (1) expresses the condition that the velocity vanishes immediately behind the surface \( \Sigma(t) \) and thus contains the requirement that the velocity be continuous over \( \Sigma(t) \), since the velocity vanishes in front of the wave \( \Sigma(t) \) by hypothesis. The pressure condition in (1) states that the normal directional derivative of the pressure vanishes on the flow side of \( \Sigma(t) \); this condition is suggested by the usual pressure condition in boundary layer theory and is applicable because of the obvious similarity between the flow in the immediate neighborhood of the surface \( \Sigma(t) \) and the flow surrounding a moving body in a viscous fluid. The last equation (1), involving the total time derivative of the temperature \( T \), expresses the condition that the temperature of a material particle has a stationary value at the instant of its contact with the rear of the surface \( \Sigma(t) \); due to the fact that the velocity vanishes on the surface \( \Sigma(t) \), this condition can also be written as

\[ \frac{\partial T}{\partial t} = 0. \quad (2) \]

We have investigated specifically the variation of the velocity of pressure shocks and also the variation of the discontinuity in the pressure over such shocks during their propagation. On the basis of the general results obtained, it is shown, briefly stated, that the successive positions of the wave surfaces \( \Sigma(t) \) in space form a family of parallel surfaces provided the velocity is constant over one of the wave fronts; for a detailed and accurate description of the results, the reader is referred to the discussion in the following article.

2. Shock and Compatibility Conditions.—Consider the general dynamical or shock conditions\(^1\)

\[ \rho(v_n - G) = \bar{\rho}(\bar{v}_n - G), \quad (3) \]

\[ [\sigma_{ij}]v^j = \rho(v_n - G)[v_i], \quad (4) \]

\[ [\sigma_{ii}]v^j + Jk[T_{ij}v^j = \rho(v_n - G)[E], \quad (5) \]

in which the usual summation convention is employed and the comma denotes differentiation with respect to the coordinates \( x^i \) of a rectangular system, assumed dynamically admissible, to which the motion of the gas is referred. The bracket denotes the difference in the values of the quantity enclosed at contiguous points on the two sides of the surface \( \Sigma(t) \); we assume, for definiteness, that the bracket stands for the value of the quantity in question immediately behind the shock surface \( \Sigma(t) \) minus its value in front of the shock. Let us also suppose that the unit normal \( v \) to the surface \( \Sigma(t) \), whose components \( v^i \) enter in the above equations, is directed into the uniform region into which the wave is propagated; then
the normal velocity $G$ of the surface $\Sigma(t)$ will have a positive value. As in the preceding section, the bar is used to denote the value of a quantity immediately behind the surface $\Sigma(t)$; thus $\bar{v}_n$ is the normal component of the velocity on the flow side of $\Sigma(t)$ while $v_n(=0)$ is the corresponding velocity in front of the surface; similarly, $\rho$ and $\bar{\rho}$ denote the values of the density on the front and rear sides of the surface $\Sigma(t)$. Other quantities occurring in the above relations are the covariant and contravariant velocity components, i.e., the $v_i$ and $v^i$, which have equal values in the rectangular system employed, the thermal conductivity $k$, the mechanical equivalent of heat $J$, the temperature $T$, the stress components $\sigma_{ij}$, and the total energy $E$ per unit mass. The latter quantities are given by the equations\(^1\)

\[
\sigma_{ij} = -p\delta_{ij} - \frac{1}{2}\mu\kappa_{k}\delta_{ij} + \mu(v_{i,j} + v_{j,i}),
\]

\[
E = \frac{1}{2}v_iv_i + Jc_{i}T + \text{const.},
\]

in which $p$ is the pressure, $\mu$ the coefficient of viscosity and $c_i$ the specific heat at constant volume; the value of the constant in (7), which depends on the selection of the zero point of energy, will not be needed in the following discussion.

Since $v_n = \bar{v}_n = 0$ and $G \neq 0$ by hypothesis, it follows from (3) that $\rho = \bar{\rho}$, i.e., the density is continuous across the surface $\Sigma(t)$. Due to the continuity of the velocity and density over $\Sigma(t)$, the compatibility conditions of the first order for these quantities have the following form\(^2\)

\[
[v_{i,j}] = \bar{v}_{i,j} = \lambda v_j; \quad \left[ \frac{\partial v_i}{\partial t} \right] = \frac{\partial \bar{v}_i}{\partial t} = -\lambda G,
\]

\[
[v_{i,j}] = \bar{v}_{i,j} = \xi v_i; \quad \left[ \frac{\partial \rho}{\partial t} \right] = \frac{\partial \bar{\rho}}{\partial t} = -\xi G,
\]

where $\lambda_i$ and $\xi$ are functions defined over the surface $\Sigma(t)$. Now $[v_i] = 0$ by hypothesis. Hence, when we combine (4) with (6) and make use of the first set of equations (8), we find that

\[
[p]v_i + \frac{1}{2}\mu\lambda_i v_j v_i = \mu\lambda_i + \mu\lambda_i v_i.
\]

Let us represent the surface $\Sigma(t)$ parametrically by functions $x^\alpha(u^1, u^2, t)$ and let us denote by $x^\alpha_\alpha$ the derivatives of the space coordinates $x^\alpha$ with respect to the parametric coordinates $u^\alpha$; we assume that $\Sigma(t)$ is regular in the sense that the functional matrix $\|x^\alpha_\alpha\|$ has rank 2 at points of this surface. Now the quantities $x^\alpha_\alpha$ for $\alpha$ fixed are the components of a vector in the space which is tangent to the surface $\Sigma(t)$. Hence, when we multiply the equations (10) by $x^\alpha_\alpha$ and sum on the repeated index $\alpha$, we find from the resulting equation that the $\lambda_i$ are the components of a vector normal to the surface. Thus, we can write

\[
\lambda_i = \lambda x_i,
\]

where $\lambda$ is a scalar function on the surface $\Sigma(t)$. It follows from the first set of equations (8) and the equations (11) that the rotation of the velocity vanishes immediately behind the surface $\Sigma(t)$, i.e., $\Sigma(t)$ is an irrotational wave. Correspondingly, we have

\[
[p] = \frac{1}{2}\mu\lambda; \quad [T, i]v^i = -\frac{\rho c_i G}{k}[T].
\]
In fact, the first equation (12) is obtained by multiplying (10) by \( \rho \), summing on the index \( i \), and making use of (11); the second equation (12) results immediately from (5) and (7) in view of the fact that the velocity vanishes on the surface \( \Sigma(t) \).

The first relation (12) provides us with an equation for the discontinuity \([p]\) in the pressure. To obtain a corresponding equation for the discontinuity \([T]\) in the temperature, we have recourse to the relation between the pressure, density, and temperature in an ideal gas, namely,

\[ p = J \rho (c_p - c_s) T, \]

in which \( c_p \) is the specific heat at constant pressure. Thus, we find that

\[ [T] = \frac{4 \mu \lambda}{3 J (c_p - c_s) \rho}; \quad [T, t] = 4 \frac{\mu G}{3 J k (\gamma - 1)}, \]

where the first of these equations is obtained immediately from the first equation (12) and the relation (13), while the second is found by elimination of \([T]\) between the second equation (12) and the first of the above equations; the quantity \( \gamma \) in the second equation (14) is the gas constant and is defined as the ratio \( c_p/c_s \) of the two specific heats, which are assumed to be constant in this theory.

We have so far used only the first of the boundary conditions (1). When account is taken of the second of these conditions, it is now readily seen that the compatibility conditions of the first order for the pressure and temperature are given by equations of the form

\[ [p, i] = \rho, i = 4 \frac{\mu g^{a \beta} \lambda_{\alpha \gamma} x_{a \beta}}{3 \mu \rho} \left[ \frac{\partial p}{\partial t} \right], \]

and

\[ [T, i] = T, i = -4 \frac{\mu G}{3 J k (\gamma - 1)} \rho + \frac{4 \mu}{3 J (c_p - c_s) \rho} g^{a \beta} \lambda_{\alpha \gamma} x_{a \beta}, \]

in which the quantities \( x_{a \beta} \) are identical with the quantities \( x_{a \beta} \) as previously defined; also the \( g^{a \beta} \) in the above equations are the contravariant components of the fundamental metric form of the surface \( \Sigma(t) \), the \( \lambda_{\alpha \gamma} \) are the surface derivatives and \( \delta / \delta t \) is the \( \delta \) time derivative of the scalar \( \lambda(\alpha, t) \) which occurs in the equations (11).

3. The Basic Differential Equations.—The differential equations for the determination of the behavior of the gas under consideration are furnished by the equation of continuity

\[ \frac{\partial p}{\partial t} + \rho \rho \rho + \rho \rho_{i, i} = 0, \]

the three second-order Navier-Stokes equations and the second-order equation for the distribution of temperature. When these equations are combined with the preceding equation (13), we have a system of six equations in the six dependent variables consisting of the pressure, density, temperature, and the three components of the velocity. The above second-order equations have not been written down explicitly since they are not needed for the derivation of the results of this article.
It follows immediately from the compatibility conditions (8) and (9) and the above equation (17) that

\[ \xi G = \rho \lambda, \]

where \( \rho \) is the constant density of the gas in front of the shock wave. Differentiating (13) with respect to \( x' \) we are led to the relations

\[ [p,] = J \rho (c_p - c_v) [T,] + J (c_p - c_v) [\rho, T]. \]

But

\[ [\rho, T] = [\rho,] [T] + T [\rho,], \]

where \( T \) is the constant temperature in the uniform region in front of the shock; these relations are easily deduced from the meaning of the brackets and the fact that the density is constant in front of the shock surface \( \Sigma(t) \). Multiplying (19) by \( \nu \) and summing on the repeated index \( i \) we now find that the resulting equation can be written in the form

\[ G^2 = \frac{3 \gamma}{4 P \rho} \left( p + \frac{4}{3} \mu \lambda \right); \quad P = \frac{\mu c_p}{k}, \]

when use is made of the equations (9), (13), (14), (15), (18), and (20); the constant \( P \) which is defined by the second equation (21) is the well-known Prandtl number.

The second term of the parenthetical expression in (21) is equal to the discontinuity \( [p] \) in the pressure from the first equation (12). Hence we can state the following result: For weak shocks, i.e., for sufficiently small values of the discontinuity \( [p] \), the velocity \( G \) of the shock is given approximately by

\[ G = \sqrt{\frac{3 \gamma p}{4 P \rho}}. \]

4. Variation in the Velocity and Strength of the Shock Wave during Propagation.—It follows immediately from the third boundary condition (2) and the last compatibility condition in (16) that

\[ \frac{c_v \lambda G^2}{k} + \frac{1}{\rho} \frac{\delta \lambda}{\delta t} = 0. \]

Now let \( \Sigma(t_0) \) represent the position of the wave surface at time \( t = t_0 \) and denote by \( \sigma \) the distance measured from \( \Sigma(t_0) \) along the normal trajectories to the family of surfaces \( \Sigma(t) \) in the direction of the propagation. The above quantity \( \lambda \) can be regarded as a function of the distance \( \sigma \) along each normal trajectory. Hence, we can write

\[ \frac{\delta \lambda}{\delta t} = G \frac{d \lambda}{d \sigma} = \frac{2 \rho PG^2}{\gamma \mu} \frac{d G}{d \sigma}, \]

when use is made of the equation obtained by differentiating (21) with respect to \( \sigma \). Substituting this expression for \( \delta \lambda/\delta t \) into (22) and eliminating the quantity \( \lambda \) by means of the relation (21), we obtain the following differential equation

\[ \frac{d G}{d \sigma} = - \frac{\rho c_v}{2k} \left( G^2 - \frac{3 \gamma p}{4 P \rho} \right) \]
for the variation of the velocity \( G \) along the normal trajectories to the family of spatial wave surfaces \( \Sigma(t) \).

If \( G_0 \) denotes the wave velocity at points of the surface \( \Sigma(t_0) \) we have

\[
G = V \left( \frac{G_0 \cosh w_\sigma + V \sinh w_\sigma}{G_0 \sinh w_\sigma + V \cosh w_\sigma} \right),
\]

(24)

by integration of (23), where

\[
V = \sqrt{\frac{3\gamma p}{4P}}, \quad w = \frac{\rho c}{2k} \sqrt{\frac{3\gamma p}{4P}}.
\]

i.e., the equation (24) gives the velocity of the wave along the normal trajectory which passes through a point on the surface \( \Sigma(t_0) \) at which the velocity has the value \( G_0 \). A corresponding expression giving the variation in the strength of the wave as measured by the discontinuity \([p]\), or by the ratio \([p]/p\), is immediately obtained by combining the first equation (12) and the equations (21) and (24). We see from (24) that

\[
G \rightarrow \sqrt{\frac{3\gamma p}{4P}}, \quad \text{as } \sigma \rightarrow \infty.
\]

It follows from this result and the equation (21) that \([p] \rightarrow 0\) as \( \sigma \rightarrow \infty \), i.e., the wave is damped out as it progresses indefinitely.

5. Waves as Parallel Surfaces.—Putting \( G = d\sigma/dt \) in (24), we obtain a differential equation for the determination of \( \sigma \) as a function of the time \( t \); the solution of this equation is given by

\[
G_0 \cosh w_\sigma + V \sinh w_\sigma = G_0 e^{(3\gamma p/8\rho) t},
\]

(25)

subject to the initial condition \( \sigma = 0 \) for \( t = 0 \). Assuming \( G_0 \) is constant over the surface \( \Sigma(t_0) \), the value of \( \sigma \) as determined from (25) will depend only on the time \( t \), i.e., the distance will be independent of the trajectory along which this distance is measured. It follows, therefore, from (24) that the velocity \( G \) must be constant over the surface \( \Sigma(t) \) at any given time \( t > t_0 \) although the value of \( G \) will depend on the spatial surface \( \Sigma(t) \). Hence, the surface derivatives \( G_\alpha = 0 \), and it follows therefore from the formula\(^2\) for the \( \delta \) time derivatives of the components \( v^i \) that \( \delta v^i/\delta t = 0 \) at all times \( t \). But this implies that the normal trajectories to the family of surfaces \( \Sigma(t) \) will be straight lines and hence these surfaces will be parallel surfaces.\(^3\)

Let us now assume, for definiteness, that the surface \( \Sigma(t_0) \), which we have supposed to be regular (see section 2), is also convex in the sense that the straight line normals issuing from this surface in the direction of the propagation do not intersect and hence form a congruence of curves in the space; this condition was assumed implicitly in the above discussion. It can then easily be shown that the successive positions of the wave surfaces \( \Sigma(t) \) in space are regular and convex surfaces. We can now state the following result. If the wave surface \( \Sigma(t) \) for \( t = t_0 \), i.e., the surface \( \Sigma(t_0) \), is a regular convex surface over which the velocity \( G \) has a constant value \( G_0 \), then the successive positions of the wave surfaces \( \Sigma(t) \) for \( t > t_0 \) will form a family of parallel surfaces, which will be regular and convex, and the wave velocity
G will have a constant value over each surface of this family. The surface $\Sigma(t)$ for $t > t_0$ can therefore be constructed from the surface $\Sigma(t_0)$ by laying off the equal distances $\sigma$ determined by (25) along the normals to $\Sigma(t_0)$, and the values of the velocity $G$ and pressure discontinuity $[p]$ over this surface $\Sigma(t)$ are readily obtained from the preceding formulæ.

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2 "Extended compatibility conditions for the study of surfaces of discontinuity in continuum mechanics," *Jour. Math. and Mech.*, 6, 311–322 (1957); also "A correction to 'Extended compatibility conditions for the study of surfaces of discontinuity in continuum mechanics,'" *ibid.*, 907–908.


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**SOME RESULTS ON THE MODULI OF RIEMANN SURFACES**

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1. In a previous paper, we showed for each integer $n \geq 2$ the existence of an algebraic variety $\mathfrak{E}'$ with the property that the points of a certain Zariski-open subset $\mathfrak{E}$ of $\mathfrak{E}'$ are in a natural one-to-one correspondence with the conformal equivalence classes of compact Riemann surfaces of genus $n$. The purpose of the present note is to extend this result by supplying answers to certain questions raised not long ago by, among others, the present author. Detailed proofs will be published later.

Specifically, our main results may be stated as follows. There exists an algebraic variety $\mathfrak{E}'$ and a regular mapping $\lambda$ of $\mathfrak{E}'$ onto $\mathfrak{E}$ such that if $e \in \mathfrak{E}$ and if $S_e$ is the Riemann surface canonically associated to $e$, then $\lambda^{-1}(e)$ is the quotient of $S_e$ by the group of all its conformal automorphisms. Moreover, the fibering $(\mathfrak{E}', \lambda, \mathfrak{E})$ is locally universal in a neighborhood of each of the points of a Zariski-open set on $\mathfrak{E}'$, which is nonempty if $n > 2$. Finally, and perhaps most interesting, $\mathfrak{E}'$, $\lambda$, and $\mathfrak{E}'$, as well as $\mathfrak{E} - \mathfrak{E}$, are all defined over the field of rational numbers.

2. In obtaining these results, we use the following notation. Let $H_n$ be the space of symmetric, complex $n \times n$ matrices $Z = X + iY$, $Y$ positive definite, $\Gamma_n$, the symplectic modular group of degree $n$, $L_Z$ the lattice in $C^n$ generated by the columns of $Z$ and of the $n \times n$ identity matrix $E$. Define an equivalence relation $L$ in $H_n \times C^n$ by $(Z, \tau) \equiv (Z', \tau')$ (mod $L$) if $Z = Z'$ and $\tau - \tau' \in L_Z$. Then $(H_n \times C^n)/L$ is a complex manifold and $\Gamma_n$ acts in a natural properly discontinuous manner on $(H_n \times C^n)/L$. If $\Gamma$ is a subgroup of $\Gamma_n$ of finite index, denote $(H_n \times C^n)/L$ by $P_\Gamma$ and $H_n/\Gamma$ by $V_\Gamma$. Let $\lambda_\Gamma$ denote the projection of $P_\Gamma$ onto $V_\Gamma$. Finally, if $A$ and $B$ are algebraic varieties and $\lambda$ is a regular mapping of $A$ onto $B$, denote by $B^0$ the diagonal of $B$, by $A^0$ the subset of $A^1$ for which $\lambda(a_1) = \cdots =