NOTE ON HYDROMAGNETIC WAVES IN A COMPRRESSIBLE
FLUID CONDUCTOR*

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1. Introduction.—Some very interesting papers have recently been published
on hydromagnetic waves. Grad\(^1\) noticed some special solutions of great signi-
cance of the linearized hydromagnetic equations when compressibility is taken
into account (all dissipative effects being neglected). If the anisotropic nature
of motion of an incompressible, inviscid, and perfectly conducting medium of
infinite extent embedded in a uniform magnetic field is strongly manifested by
undistorted propagation of waves in one direction only—along magnetic lines
of force—the compressibility acts in an even stranger manner. This becomes
immediately apparent when one specifies the disturbance by means of vorticity
and current density; then, the longitudinal components only of both vorticity
and current density propagate along magnetic lines of force at the Alfvén velocity
\[ A_0 = \frac{H_0 \sqrt{\mu_0}}{4\pi \rho_0}, \]
where \( H_0 \) is the magnitude of the magnetic field \( \mathbf{H} \) and \( \mu_0 \)
and \( \rho_0 \) the permeability and the density of the medium in the unperturbed state
(emu units). This fascinating property attracted this author's attention since
he began his studies on magnetohydrodynamics. It appears in Grad's article\(^1\)
and an earlier paper\(^2\) by the writer. Part of this theory can be found in earlier
papers by Baños\(^3\) (where a complete up-to-date bibliography on that subject is
given).

We also mention Ludford's contribution\(^4\) to the subject, but this is more related
to a paper\(^5\) on "Dissipative Effects in Magnetohydrodynamic Waves" to be
published in the Comptes Rendus.

Scholte\(^6\) has put forward a similar theory in order to explain giant geomagnetic
pulsations such as that recorded in 1958 in Europe and analyzed by Veldkamp.\(^7\)
The observed giant pulsations, it is argued, are mainly caused by the rotational
part of the primary disturbance which propagates along geomagnetic lines of
force without any great loss of energy. Therefore, this theory seems to offer, at
least tentatively, an explanation of ionospheric noise generation and geomagnetic
pulsations and may apply to radiations from natural and artificial sources, but
there is, of course, considerably more study effort to be done in this area.

This was about the state of art at the beginning of 1960 when two striking
papers by Lighthill\(^8\) revitalized the theory and its potential applications. Cer-
tainly, the beautiful simplicity of certain wave-motion equations could not escape
Lighthill's attention who, in fact, rediscovered some important features\(^1-7\) of this
theory in the course of his "Studies on Magnetohydrodynamic Waves and Other
Anisotropic Motions.\(^8\)

The purpose of the present paper is first to improve certain results obtained in
our earlier paper\(^2\) and second, in the light of these results, to give a brief systematic
account of wave-motion equations satisfied by the rate-of-deformation \( \epsilon_{ij} \) and a
new pseudo-tensor \( \gamma_{ij} \), the latter being derived from the magnetic field in the
same manner as the former is derived from material velocity; $\gamma_{ij}$ appears somewhat coupled to $e_{ij}$ in our hydromagnetic formulation. Lighthill was the first to give a wave equation for the $e_{33}$ component of $e_{ij}$ and the expansion (or first deformation invariant), while Grad gave similar equations (coupled) for expansion, density, and longitudinal components of material velocity and magnetic induction. But up to now, there has been no systematic attempt to formulate wave-motion equations for $e_{ij}$ and $\gamma_{ij}$. There are precisely such equations that I will derive in Section 6.

2. Fundamental Equations.—We begin with the following linearized hydromagnetic equations, which, neglecting all dissipative effects, are

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla \phi + \frac{\mu_0 H_0}{4\pi} \frac{\partial \mathbf{h}}{\partial z},$$

$$\frac{\partial \mathbf{h}}{\partial t} = H_0 \frac{\partial \mathbf{v}}{\partial z} - H_0 \nabla \cdot \mathbf{v},$$

$$\frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot \mathbf{v} = 0,$$

$$\nabla \cdot \mathbf{h} = 0,$$

where the $z$-axis is taken in the direction of field $H_0$, $\rho, \mathbf{v}, \mathbf{h}$ denote the perturbations in density $\rho_0, \mathbf{v}_0 = 0$, $H_0$, and $\phi = a_0^2 \rho + (\mu_0 H_0 h_z)/4\pi$, where $a_0$ is the ordinary sound speed in the unperturbed state.

Taking the curl of equations (1) and (2), we have

$$2 \rho_0 \frac{\partial \omega}{\partial t} = \mu_0 H_0 \frac{\partial j}{\partial z},$$

$$4\pi \frac{\partial j}{\partial t} = 2H_0 \frac{\partial \omega}{\partial z} - \frac{H_0}{\rho_0} \times \nabla \frac{\partial \rho}{\partial t},$$

where $\omega = (1/2) \text{curl} \mathbf{v}$ is the vorticity, and $j = (1/4\pi) \text{curl} \mathbf{h}$ is the current density.

It follows that vorticity components $\omega_x, \omega_y, \omega_z$ and current-density components $j_x, j_y, j_z$ satisfy the following wave-motion equations:

$$\frac{\partial^2 \omega_x}{\partial t^2} - A_0^2 \frac{\partial^2 \omega_x}{\partial z^2} = \frac{A_0^2}{2\rho_0} \frac{\partial^2 \rho}{\partial t \partial y \partial z},$$

$$\frac{\partial^2 \omega_y}{\partial t^2} - A_0^2 \frac{\partial^2 \omega_y}{\partial z^2} = -\frac{A_0^2}{2\rho_0} \frac{\partial^2 \rho}{\partial t \partial x \partial z},$$

$$\frac{\partial^2 \omega_z}{\partial t^2} - A_0^2 \frac{\partial^2 \omega_z}{\partial z^2} = 0,$$

and

$$\frac{\partial^2 j_x}{\partial t^2} - A_0^2 \frac{\partial^2 j_x}{\partial z^2} = \frac{H_0}{4\pi \rho_0} \frac{\partial^2 \rho}{\partial t \partial y \partial z},$$

$$\frac{\partial^2 j_y}{\partial t^2} - A_0^2 \frac{\partial^2 j_y}{\partial z^2} = -\frac{H_0}{4\pi \rho_0} \frac{\partial^2 \rho}{\partial t \partial x \partial z}.$$
\[
\frac{\partial^2 j_z}{\partial t^2} - A_0^2 \frac{\partial^2 j_z}{\partial x^2} = 0,
\]
(10)

where
\[
j_z = \pm \frac{H_0}{2\pi A_0} \omega_z.
\]
(11)

Therefore, it appears that while the transverse components are coupled to the density oscillations and are convected with the fluid, the longitudinal components ignore these density oscillations and are propagated along lines of force (in both directions) without attenuation.

The coupling relationship (11) between longitudinal components shows that

(a) it does not depend on the magnitude of the magnetic field present;
(b) the vanishing of either component involves the vanishing of the other; this occurs when either quantity is zero initially.

The density satisfies\textsuperscript{1,2}
\[
\frac{\partial^2 \rho}{\partial t^2} = a_0^2 \nabla_z^2 \rho + \frac{A_0^2 \rho_0}{H_0} \nabla_z h_z = a_0^2 \nabla_z^2 \rho + \frac{4\pi A_0^2 \rho_0}{H_0} \left( \frac{\partial j_z}{\partial y} - \frac{\partial j_y}{\partial x} \right),
\]
(12)

where \( \nabla_z^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \).

3. Wave-Motion Equations for the Density and Transverse Components.—These results can be improved as follows: First, elimination of \( j_z \) and \( j_y \) between equations (9) and (12) yields
\[
\frac{\partial^2 \rho}{\partial t^2} \left( \frac{\partial^2 \rho}{\partial t^2} - a_0^2 \nabla_z^2 \rho \right) - A_0^2 \frac{\partial^2}{\partial z^2} \left( \frac{\partial^2 \rho}{\partial t^2} - a_0^2 \nabla_z^2 \rho \right) = A_0^2 \frac{\partial^2}{\partial t^2} \left( \frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} \right),
\]
(13)

where a two-dimensional Laplacian appears. This equation shows plainly the radical departure of the "new sound-wave equation" from the ordinary sound-wave equation in the absence of a magnetic field. It can be used to actually evaluate \( \rho \) by successive approximations (Picard reiteration method). Equation (13) can be rewritten
\[
\frac{\partial^2}{\partial t^2} \left( \frac{\partial^2 \rho}{\partial t^2} \right) - \left( a_0^2 + A_0^2 \right) \nabla_z^2 \rho + a_0^2 A_0^2 \frac{\partial^2}{\partial z^2} \nabla_z^2 \rho = 0.
\]
(14)

Differentiation with respect to \( t \) of terms of equation (14) yields an equation given by Lighthill for the expansion \( \Delta = \text{div} \, \mathbf{v} \). For time harmonic dependence, one finds a result given earlier by Bafios (see reference 3b, p. 352).

On the other hand, elimination of \( \rho \) between equations (7) and (14) gives
\[
\frac{\partial^2}{\partial t^2} \left[ \frac{\partial^2}{\partial t^2} \left( \frac{\partial^2}{\partial t^2} \right) - (a_0^2 + A_0^2) \nabla_z^2 \right] + a_0^2 A_0^2 \frac{\partial^2}{\partial z^2} \nabla_z^2 \left( \frac{\partial^2}{\partial t^2} \right) - A_0^2 \frac{\partial^2}{\partial z^2} \left( \frac{\partial^2}{\partial t^2} \left( \frac{\partial^2}{\partial t^2} \right) - (a_0^2 + A_0^2) \nabla_z^2 \right) = 0.
\]
(15)

Similarly, elimination of \( \rho \) between equations (9) and (14) gives
\[
\left\{ \frac{\partial^2}{\partial t^2} \left[ \frac{\partial^2}{\partial t^2} - (a_0^2 + A_0^2) \nabla^2 \right] + a_0^2 A_0^2 \frac{\partial^2}{\partial x^2} \nabla^2 \right\} - \\
A_0^2 \frac{\partial^2}{\partial x^2} \left[ \frac{\partial^2}{\partial t^2} - (a_0^2 + A_0^2) \nabla^2 \right] + a_0^2 A_0^2 \frac{\partial^2}{\partial x^2} \nabla^2 \right\} \right\}_{j_x}^{j_y} = 0. \quad (16)
\]

Equations (15) and (16) show that the quantities
\[
\left[ \frac{\partial^2}{\partial t^2} - (a_0^2 + A_0^2) \nabla^2 \right] + a_0^2 A_0^2 \frac{\partial^2}{\partial x^2} \nabla^2 \right\}_{j_x, j_y} \quad (17)
\]
are propagated along magnetic lines of force at Alfvén velocity \(A_0\). These quantities are identically zero if they are zero initially; under this condition, equations (15) and (16) reduce to fourth-order equations of the same type as equation (14).

4. Coupling Relationship between Vorticity and Current Density.—Following a suggestion due to C. Walén (personal communication), we can put
\[
j = \pm \frac{H_0}{2 \pi A_0} \omega + H_0 \times \text{grad} \varphi, \quad (18)
\]
for \(\text{div} (H_0 \times \text{grad} \varphi)\) is identically zero and relation (11) is satisfied. To determine \(\varphi\), we substitute (18) into equations (5) and (6). This gives
\[
H_i \times \text{grad} \left( \frac{\partial \varphi}{\partial t} \pm A_0 \frac{\partial \varphi}{\partial x} + \frac{1}{4 \pi \rho_0} \frac{\partial \rho}{\partial t} \right) = 0. \quad (19)
\]
Hence, \(\varphi\) satisfies
\[
\frac{\partial \varphi}{\partial t} \pm A_0 \frac{\partial \varphi}{\partial x} + \frac{1}{4 \pi \rho_0} \frac{\partial \rho}{\partial t} = 0. \quad (20)
\]
Elimination of \(\rho\) between equations (14) and (20) gives
\[
\frac{\partial}{\partial t} \left[ \frac{\partial^2 \varphi}{\partial t^2} - (a_0^2 + A_0^2) \nabla^2 \varphi \right] + a_0^2 A_0^2 \frac{\partial^2}{\partial x^2} \nabla^2 \varphi \right] \pm \\\nA_0 \frac{\partial}{\partial x} \left[ \frac{\partial^2 \varphi}{\partial t^2} - (a_0^2 + A_0^2) \nabla^2 \varphi \right] + a_0^2 A_0^2 \frac{\partial^2}{\partial x^2} \nabla^2 \varphi \right] = 0, \quad (21)
\]
as one would have expected by virtue of equations (15) and (16).

5. A New Pseudo-Tensor, \(\gamma_{ij}\), Derived from Magnetic Field.—By the way of digression, we now introduce the quantity
\[
\gamma_{ij} = \frac{1}{2} \left( \frac{\partial h_i}{\partial x_j} + \frac{\partial h_j}{\partial x_i} \right), \quad (22)
\]
where for convenience usual tensor notations are employed. This is obviously mathematically derived from the magnetic field in the same way as the rate-of-deformation
\[
e_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \quad (23)
\]
is derived from material velocity. \(\gamma_{ij}\) is somewhat coupled to \(e_{ij}\) by our hydro-
magnetic formulation, equations (1)–(4), as will shortly appear. Its physical
significance is not obvious (it has dimensions of current density), but, following
an analysis for material velocity due to Stokes, we can say this: if one compares,
at the time $t$, the magnetic fields at two infinitely near material particles at $x_i$ and
$x'_i = x_i + \delta x_i$, then one has

$$h'_i = h_i + \frac{\partial h_i}{\partial x_j} \delta x_j,$$  \hspace{1cm} (24)

which can be written as

$$h'_i = h_i + \frac{1}{2} \left( \frac{\partial h_i}{\partial x_j} - \frac{\partial h_j}{\partial x_i} \right) \delta x_j + \frac{1}{2} \left( \frac{\partial h_i}{\partial x_j} + \frac{\partial h_j}{\partial x_i} \right) \delta x_j$$  \hspace{1cm} (25)

or, in vector notations,

$$\mathbf{h'} = \mathbf{h} + 2\pi \mathbf{j} \times \delta \mathbf{r} + (1/2) \text{Grad}_x F,$$  \hspace{1cm} (26)

where we set

$$F = h_i \delta x_i \delta x_j,$$  \hspace{1cm} (27)

and the gradient is taken with respect to $\delta x_i$ ($x_i$ being fixed). Now, while the
second term on the right side of equation (26) expresses plainly Laplace’s law,
the last part is somewhat more difficult to interpret and is arrested here for further
investigation. With these preliminaries, we pass to the last section of this paper
and give wave equations for both $e_{ij}$ and $\gamma_{ij}$.

6. Wave-Motion Equations for $e_{ij}$ and $\gamma_{ij}$.—When $e_{ij}$ and $\gamma_{ij}$ are introduced
in equations (1) and (2), these become

$$\rho \frac{\partial e_{11}}{\partial t} = -a_0^2 \frac{\partial^2 \rho}{\partial x^2} + \mu_0 H_0 \frac{\partial j_y}{\partial x},$$

$$\rho \frac{\partial e_{22}}{\partial t} = -a_0^2 \frac{\partial^2 \rho}{\partial y^2} - \mu_0 H_0 \frac{\partial j_z}{\partial y},$$

$$\rho \frac{\partial e_{33}}{\partial t} = -a_0^2 \frac{\partial^2 \rho}{\partial z^2},$$

$$\rho \frac{\partial e_{23}}{\partial t} = -a_0^2 \frac{\partial^2 \rho}{\partial y \partial z} - \frac{\mu_0 H_0}{2} \frac{\partial j_z}{\partial z},$$

$$\rho \frac{\partial e_{31}}{\partial t} = -a_0^2 \frac{\partial^2 \rho}{\partial z \partial x} + \frac{\mu_0 H_0}{2} \frac{\partial j_y}{\partial y},$$

$$\rho \frac{\partial e_{12}}{\partial t} = -a_0^2 \frac{\partial^2 \rho}{\partial x \partial y} + \frac{\mu_0 H_0}{2} \left( \frac{\partial j_y}{\partial y} - \frac{\partial j_z}{\partial z} \right).$$  \hspace{1cm} (28)

These equations are highly unsymmetrical and show perhaps more plainly than
equations (1) and (2) the strongly anisotropic character of the electrically con-
ducting fluid. However, using the results obtained in Section 3, the conclusions
to be drawn are surprisingly simple. First, it follows at once by virtue of equation
(14), that $e_{33}$ satisfies
and

\[
\frac{\partial \gamma_{11}}{\partial t} = H_0 \frac{\partial \delta_{11}}{\partial \xi},
\]

\[
\frac{\partial \gamma_{21}}{\partial t} = H_0 \frac{\partial \delta_{21}}{\partial \xi},
\]

\[
\frac{\partial \gamma_{22}}{\partial t} = \frac{H_0}{\rho_0} \frac{\partial \delta_{22}}{\partial \xi} + H_0 \frac{\partial \delta_{22}}{\partial \zeta},
\]

\[
\frac{\partial \gamma_{31}}{\partial t} = \frac{H_0}{2} \frac{\partial \delta_{31}}{\partial \xi} + H_0 \frac{\partial \delta_{31}}{\partial \zeta},
\]

\[
\frac{\partial \gamma_{32}}{\partial t} = \frac{H_0}{2} \frac{\partial \delta_{32}}{\partial \xi} + H_0 \frac{\partial \delta_{32}}{\partial \zeta},
\]

\[
\frac{\partial \gamma_{33}}{\partial t} = \frac{H_0}{\rho_0} \frac{\partial \delta_{33}}{\partial \xi} + H_0 \frac{\partial \delta_{33}}{\partial \zeta},
\]

\[
\frac{\partial^2 \delta_{33}}{\partial t^2} \left( \frac{\partial^2 \delta_{33}}{\partial \xi^2} - (a_0^2 + A_0^2) \nabla^2 \delta_{33} \right) + a_0^2 A_0^2 \frac{\partial^2 \delta_{33}}{\partial \zeta^2} \nabla^2 \delta_{33} = 0,
\]

(30)
a result obtained earlier by Lighthill.⁸ We may add that the same equation is also satisfied by \( \gamma_{33} \), as seen by inspection of the third equation in (29).

By successive eliminations, we then find that all other components satisfy a unique sixth-order differential equation of the type as that encountered in Section 3; therefore, we have

\[
\left\{ \frac{\partial^2}{\partial t^2} \left[ \frac{\partial^2}{\partial \xi^2} \left( \frac{\partial^2}{\partial \xi^2} - (a_0^2 + A_0^2) \nabla^2 \delta \right) + a_0^2 A_0^2 \frac{\partial^2}{\partial \zeta^2} \nabla^2 \right] \right\} \gamma_{ij} = 0,
\]

(31)
where we use the prime in order to exclude the values \( i = j = 3 \) of suffixes; for \( i = j = 3 \), we have equation (30).

If one puts

\[
e^{i\omega} = \exp[i(\omega t - \alpha x - \beta y - \gamma z)],
\]

(32)
once has

\[
(\omega^2 - A_0^2 \gamma^2)[(\omega^2 - a_0^2 \gamma^2)(\omega^2 - A_0^2 \gamma^2) - (\alpha^2 + \beta^2)((a_0^2 + A_0^2) \omega^2 - a_0^2 A_0^2 \gamma^2)] = 0.
\]

(33)
Hence: Either

(a) \( \gamma = \pm \omega/A_0 \) and then \( \gamma'_{ij}, \gamma'_{ij} \) are propagated solely along magnetic lines of force without attenuation at the Alfvén velocity \( A_0 \); or

(b) \( \gamma = \pm \omega/A_0 \), and then

\[
\alpha^2 + \beta^2 = \frac{(\omega^2 - a_0^2 \gamma^2)(\omega^2 - A_0^2 \gamma^2)}{(a_0^2 + A_0^2) \omega^2 - a_0^2 A_0^2 \gamma^2},
\]

(34)
which represents the wave number surface given by Lighthill.8

Under (b), we distinguish two subcases:

(b.1) \( \gamma = \pm \omega / a_0 \), and then \( e'_i, \gamma'_i \) are propagated one-dimensionally, along magnetic lines of force, at the sound speed \( a_0 \); in this case, \( \alpha = \beta = 0 \), that is, the ovoid sheet of (34) collapses into the \( \gamma \)-axis.

(b.2) \( \gamma \neq \pm \omega / a_0 \); by rewriting equation (34) as follows,

\[
\alpha^2 + \beta^2 = \frac{1}{A_0^2} \left( \frac{\omega^2 - a_0^2 \gamma^2}{\omega^2 - A_0^2 \gamma^2} \right)
\]

or

\[
\alpha^2 + \beta^2 = \frac{1}{a_0^2} \left( \frac{\omega^2 - a_0^2 \gamma^2}{\omega^2 - A_0^2 \gamma^2} \right)
\]

two limiting cases, considered in slightly different forms by Bānos,3b Grad,1 and Lighthill,8 are readily revealed:

(b.2.1) \( A_0 \gg a_0 \), when equation (35) reduces approximately to

\[
\alpha^2 + \beta^2 + \gamma^2 = \frac{\omega^2}{A_0^2}
\]

and we have propagation in all directions with nearly equal velocity \( A_0 \);

(b.2.2) \( A_0 \ll a_0 \), when equation (36) reduces approximately to

\[
\alpha^2 + \beta^2 + \gamma^2 = \frac{\omega^2}{a_0^2}
\]

and we have isotropic tri-dimensional propagation with velocity nearly \( a_0 \).

The singular case \( \gamma = \pm \omega \sqrt{a_0^{-2} + A_0^{-2}} \) when \( \alpha^2 + \beta^2 = \infty \), requires more care and is omitted here.

Appendix.—The wave-motion equations for \( e_i \) and \( \gamma_i \) may also be derived by the following brief analysis, which re-emphasizes the compressibility effect of discrimination between longitudinal and transverse components.

The \( z \)-component of material velocity satisfies

\[
\rho_0 \frac{\partial v_z}{\partial t} = -a_0^2 \frac{\partial \rho}{\partial z}.
\]

Hence, by virtue of equation (14), we have at once

\[
\left[ \frac{\partial^2}{\partial t^2} \left( \frac{\partial^2}{\partial t^2} - (a_0^2 + A_0^2) \nabla^2 \right) + a_0^2 A_0^2 \frac{\partial^2}{\partial z^2} \right] v_z = 0.
\]

On the other hand,

\[
\frac{\partial h_z}{\partial t} = H_0 \frac{\partial v_z}{\partial z} - H_0 \text{ div } \mathbf{v}.
\]

Since both \( v_z \) and \( \text{div } \mathbf{v} \) satisfy (A.2), it follows that
Combining equations for the transverse components in (1) and (2) with equations (14), (A.2), and (A.4), we obtain

\[
\left[ \frac{\partial^2}{\partial t^2} \left( \frac{\partial^2}{\partial t^2} - (a_0^2 + A_0^2)\nabla^2 \right) + a_0^2 A_0^2 \frac{\partial^2}{\partial z^2} \nabla^2 \right] h_z = 0. \quad (A.4)
\]

The wave-motion equations for \( \epsilon_{ij} \) and \( \gamma_{ij} \) become obvious. Furthermore, the propagation of transverse divergence

\[
\delta = \frac{\partial \epsilon_{ij}}{\partial x} + \frac{\partial \gamma_{ij}}{\partial y},
\]

as well as that of transverse divergence of magnetic field, also becomes apparent. According to Lighthill (see reference 8b, p. 467), the propagation of \( v_z \) and \( \delta \), governed by equations (A.2) and (A.5) respectively, are important for waves through gases at pressures small compared with magnetic pressure.

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2 Carstoiu, J., these PROCEEDINGS, 46, 131–136 (1960).