CONVOLUTION OPERATORS ON BANACH SPACE VALUED FUNCTIONS

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Communicated by A. Zygmund, January 2, 1962

The purpose of this paper is to obtain systematically certain classical inequalities concerning the Hilbert transform, the function $g$ of Littlewood and Paley, their generalizations to several variables, and related results. This we accomplish by establishing certain inequalities for convolution operators on Banach space valued functions.

Given a Banach space $B$, $|f|$ will denote the norm of the element $f$ of $B$; $M(B)$ will denote the space of strongly measurable $B$-valued functions $f(x)$ defined in $E_n$; $L^p(B)$ will denote the class of bounded functions in $M(B)$ with compact support, and $L^p(B)$ the space of functions $f(x)$ in $M(B)$ such that $|f(x)|$ belongs to $L^p$, the norm of a function of $L^p(B)$ will be denoted by $||f||_p$. Given a function $f$ in $M(B)$, $E_s(f)$ will stand for the measure of the set of points where $|f| > t$. Finally, $c$ will always denote a constant, and $c$ a fixed multiple of $c$.

**Lemma 1.** Let $A$ be a linear operator defined in $L^p_0(B_1)$ with values in $M(B_2)$. Suppose that

$$E_s(Af) \leq c t^{-r}||f||_r, \quad E_s(Af) \leq c d^{-r}||f||_p, \quad r > 1.$$  

Then for each $p$, $1 < p < r$, we have $Af \in L^p(B_2)$ and $||Af||_p \leq c_0||f||_p$, where $c_0$ depends only on $c_1$, $c_2$, and $p$.

**Proof:** Let $n(x) = f(x)|f(x)|^{-1}$ if $f(x) \neq 0$, $n(x) = 0$ otherwise. Consider the operation $Bg = \left|A[n(x)]\right|$, where $g$ is a numerical function. Then, clearly, $B$ is sublinear and

$$E_s(Bg) \leq c t^{-r}||g||_r; \quad E_s(Bg) \leq c d^{-r}||g||_p.$$  

Thus, by the theorem of Marcinkiewicz, we have $||Bg||_p \leq c_p||g||_p$ with $c_p$ depending only on $c_1$, $c_2$, and $p$. Setting $g = |f(x)|$, we obtain the desired result.

**Lemma 2.** Let $f(x)$ be a real-valued nonnegative measurable bounded function with compact support in $E_n$, and $t > 0$. Then there exists a family of nonoverlapping cubes $Q_n$ such that

$$t \leq |Q_n|^{-1} \int_{Q_n} f(x) dx \leq 2^t t$$  

and $f(x) \leq t$ almost everywhere outside $\bigcup Q_n$.

This result is well known.

**Theorem 1.** Let $A$ be a linear operator defined in $L^p_0(B_1)$ with values in $M(B_2)$ such that

(i) $E_s(Af) \leq c t^{-r}||f||_r$; for some $r > 1$. 

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Suppose that $B$, to coincide with class of values in norms be can $n$ for $E,(Af)$, it follows with $\tau rf$. Since $I|f|t1$ measure. Thus, $E,1h12(Ag)$, $\ldots$ $x_n)$.

Let now $D_1$ be the union of all spheres $S_n$ concentric with $Q_n$ and with diameter equal to $c_2$ times the diameter of $Q_n$. Let $h_n$ be the restriction of $h$ to $Q_n$. Since $\sum h_n$ converges to $h$ in $L^r(B_1)$, condition (i) implies that $A(\sum h_n) \rightarrow Ah$ in measure. Thus, $|Ah| \leq \sum |Ah_n|$, and by (ii)

$$\int D_1|Ah|\,dx \leq \sum \int D_1|Ah_n|\,dx \leq \sum \int S_n^c|Ah_n|\,dx \leq c_3 \sum \|h_n\|_1 \leq c_6\|h\|_1,$$ 

where $D_1'$ and $S_n'$ are the complements of $D_1$ and $S_n$ respectively. Consequently,

$$E_i\gamma(Ah) \leq |D_1| + 2t^{-1}\int D_1|Ah|\,dx \leq \delta t^n|D| + 2t^{-1}c_3\|h\|_1.$$ 

Since $\|h\|_1 \leq 2\|f\|_1$ and

$$|D| = \sum |Q_n| \leq \sum t^{-1}\int Q_n|f|\,dx \leq t^{-1}\|f\|_1$$

it follows that $E_i\gamma(Ah) \leq (\delta t^n + 4c_3)t^{-1}\|f\|_1$. Hence,

$$E_i(Af) = E_i(Ag + Ah) \leq E_i\gamma(Ag) + E_i\gamma(Ah) \leq (2^n(r - 1) + \delta t^n + 4c_3) t^{-1}\|f\|_1$$

and the desired result is established.

Given a point $x = (x_1, x_2, \ldots, x_n)$ in $E_n$ we denote by $\bar{x}$ its projection $\bar{x} = (x_2, \ldots, x_n)$ on $E_{n-1}$. With a function $f(x)$ in $M(B)$, we associate the function $F(\bar{x}) = \pi f$ with values in $M(B)$ of the real line defined by $F(\bar{x}) = f(\cdot, x_2, \ldots, x_n)$. Now we introduce norms with multiple exponents $P = (p_1, p_2, \ldots, p_n)$ in $M(B)$ (see (1)). For $n = 1$, these norms coincide with the ordinary $L^p(B)$ norms. For $n > 1$, they can be defined by induction by $\|f\|_P = \|\pi f\|_P$, where $\pi f$ is regarded as a function with values in $L^p(B)$ of the real line and $\pi f = (p_1, p_2, \ldots, p_n)$. The space $L^p(B)$ is the class of functions in $M(B)$ for which $\|f\|_P < \infty$. If $P = (p, p, \ldots, p)$ then $L^p(B)$ coincides with $L^p(B)$ and $\|f\|_P = \|f\|_P$.

THEOREM 2. Let $k(x)$ be a function whose values are bounded linear operators from $B_1$ to $B_2$; let $k$ be measurable and integrable on compact sets. For $f \in L^p_0(B_1)$, define

$$Af = \int f(x - y) f(y) dy.$$ 

Suppose that

(i) for some $r > 1$ and all $f$ the inequality $\|Af\|_r \leq c_0\|f\|_r$ holds, and
(ii) for all \( y \),
\[
\int_{|x| > 4lq} |k(x - y) - k(x)| \, dx \leq c_r.
\]

Then \( Af \in L^p(B_2) \) for all \( P = (p_1, p_2 \ldots p_n), 1 < p_i < \infty \), and \( ||Af||_p \leq c_r||f||_p \) with \( c_r \) depending only on \( c_1, c_2, \) and \( P \).

**Proof:** Let \( \tilde{A} \) be the operator on \( L^0_0(B_1^*) \) defined by
\[
\int k^*(y - x)g(y)dy.
\]

It is easy to see that the inequality \( ||A\gamma||_q \leq c||\gamma||_q \) holds if and only if \( ||Af||_p \leq c||f||_p \) valid. Furthermore, since \( |k^*| = |k|, \tilde{A} \) satisfies the same assumptions as \( A \) does with exponent \( r/(r - 1) \). Now suppose that \( f \) supports in \( |x - x_0| \leq \rho \) and mean value zero. Then by (ii)
\[
\int k^*(y - x)g(y)dy = \int \int k(x - y)f(y)dy dx = \int \int k(x - y)f(y - x_0)dy dx \leq \int \int k(x - y)dy dx \leq c||f||_1.
\]

Thus \( A \) satisfies the assumptions of Theorem 1 and consequently \( ||Af||_p \leq c_r||f||_p \) and \( ||A\gamma||_q \leq c_r||\gamma||_q \) for \( 1 < p < r \) and \( q = r/(p - 1) \). On the other hand \( \tilde{A} \) also satisfies the assumptions of Theorem 1 which implies that \( ||\tilde{A}\gamma||_q \leq c_r||\gamma||_q \) and \( ||Af||_p \leq c_r||f||_p \) for \( 1 < q < r/(r - 1) \) and \( p = q/(q - 1) \). Hence, \( ||Af||_p \leq c_\gamma||f||_p \) for all \( p, 1 < p < \infty \).

To prove this inequality for norms with multiple exponents, we argue by induction. Assume the theorem proved for functions of \( n - 1 \) variables. Let \( M(B, E_0^0, L^0_0(B, E_m)) \) denote the spaces of B-valued functions on \( E_m \) which are measurable, etc. Let \( P = (p_1, p_2 \ldots p_n), 1 < p_i < \infty \), be given. Set \( B_1 = L^p_0(B_1, E_1) \) and \( B_2 = L^p(B_2, E_2) \), and let \( \pi \) be the operation introduced above. Then \( ||\pi f||_p = ||f||_p \) and \( ||\pi f||_p = ||f||_p \). Furthermore, \( \pi \) maps \( L^0_0(B_1, E_m) \) into \( L^0_0(B_1, E_{n-1}) \). If \( f \in M(B, E_0) \), define \( \gamma f \) by \( \gamma f(x_1, x_2 \ldots x_n) = f(x_1, \ldots x_n) \) if \( |x_1| < r \) and \( \gamma f \) otherwise. Then \( K_0(\tilde{x}, \tilde{y}) \) denotes the norm of \( K_0 \), as an operator on \( B_1 \) we have
\[
|K_0(\tilde{x})| \leq \int_0^{2r} K_0(t, x_2, \ldots x_n) \, dt
\]
which shows that \( K_0(\tilde{x}) \) is locally integrable in \( E_{n-1} \). Furthermore,
\[
\int_{|\tilde{x}| > 4l} |K_0(\tilde{x} - \tilde{y}) - K_0(\tilde{x})| \, d\tilde{y} \leq \int |K_0(\tilde{x} - 2\tilde{y}) - K_0(\tilde{x})| \, d\tilde{y} \leq c_0.
\]

Now, for \( F \in L^0_0(B_1, E_{n-1}) \) define
\[
\mathfrak{K}_G F = \int K_0(\tilde{x} - \tilde{y})F(y)d\tilde{y}.
\]

Then, as readily seen, \( \pi \gamma A \gamma f = \mathfrak{K}_G F \). Thus,
\[ \| \gamma \sigma f \|_p = \| \pi \gamma \sigma A f \|_p = \| \gamma \sigma A f \|_p \leq c_p \| \gamma f \|_p \leq c_p \| \sigma f \|_p = c_p \| \sigma f \|_p \]

and consequently \( \| \gamma \sigma f \|_p \leq c_p \| \sigma f \|_p \) for all \( F \) of the form \( F = \pi f, f \in L^p(B_1, E_u) \).

Now, functions of this form are dense in \( L^p(B_1, E_u) \) with respect to the norm of \( P(B_1, E_u) \) and a passage to the limit yields \( \| \gamma \sigma f \|_p \leq c_p \| \sigma f \|_p \) for all \( F \in L^p(B_1, E_u) \). This and the inequalities for \( K_s(x) \) obtained above show that \( \gamma \sigma \) satisfies the assumptions of the theorem uniformly in \( v \), and our inductive hypothesis yields \( \| \gamma \sigma f \|_p \leq c_p \| \sigma f \|_p \) for all \( F \in L^p(B_1, E_u) \) and \( \| \sigma f \|_p \) is the norm of \( \sigma f \) in \( L^p(B_1, E_u) \), and \( c_p \) is independent of \( v \).

Let now \( f \) be an element of \( L^p(B_1, E_u) \). Then,

\[ \| \gamma \sigma f \|_p = \| \pi \gamma \sigma A f \|_p = \| \pi \sigma f \|_p = c_p \| \sigma f \|_p. \]

Since \( \gamma \sigma f \sigma f \) for \( v \) sufficiently large we have

\[ \| \sigma f \|_p = \lim_{v \to \infty} \| \gamma \sigma f \|_p \leq c_p \| \sigma f \|_p \]

and the theorem is established.

**Remark:** It is readily seen that the theorem is still valid for \( P = (p, p, \ldots, p) \), \( 1 < p < \infty \), if condition (ii) is replaced by the weaker one: for all \( u \in B_1 \) and \( v \in B_2^* \)

\[ \begin{align*}
\int_{|x| > \frac{1}{|u|}} |k(x - y) - k(x)| u |dx| & \leq c_2 |u| ; \\
\int_{|x| > \frac{1}{|v|}} |k^*(x - y) - k^*(x)| v |dx| & \leq c_2 |v| .
\end{align*} \]

**Theorem 3.** Let \( k(x) \) be a function whose values are bounded operators from the Hilbert space \( H_1 \) to the Hilbert space \( H_2 \); let \( k \) be measurable and integrable on compact sets not containing the point \( x = 0 \). Suppose that

(i) for \( 0 < \epsilon < \delta \),

\[ \left| \int_{|x| < \epsilon} k(x) dx \right| \leq c \]

and for each \( u \in H_1 \),

\[ \int_{|x| < \epsilon} |k(x)| u |dx| \leq c_p \]

(ii) for \( u \in H_1 \),

\[ \int_{|x| > \frac{1}{|u|}} |k(x)| u |dx| \leq c \]

(iii) for \( u \in H_1 \) and \( v \in H_2 \),

\[ \int_{|x| > \frac{1}{|u|}} |k(x) - k(x)| u |dx| \leq c |u| ; \\
\int_{|x| > \frac{1}{|v|}} |k^*(x - y) - k^*(x)| v |dx| \leq c |v| .
\]

Let \( A \) be the operator on \( L^p_p(H_1) \) with values in \( M(H_2) \) defined by

\[ A f = \int_{|x| > \epsilon} k(x - y) f(y) dy . \]

Then if \( 1 < p < \infty \) we have \( \| A f \|_p \leq c_p \| f \|_p \) with \( c_p \) depending only on \( c \) and \( p \). Furthermore, \( A f \) converges in \( L^p(H_2) \) as \( \epsilon \to 0 \). If instead of (iii) the stronger condition
\[ \int_{|x| > 4|y|} |k(x - y) - k(x)| \, dx \leq c \]

is satisfied, the same conclusion holds with \( p \) replaced by \( P = (p_1, p_2, \ldots, p_n) \), \( l < p_i < \infty \).

**Proof.** Let \( 4|y| < \rho \) and \( u \in H_1 \). Then,

\[
\int_{|x| \leq 2\rho} [k(x - y) - k(x)] \, dx \leq \int_{|x| \leq 2\rho} \left| k(x - y) - k(x) \right| \, dx + \int_{|x| \leq 2\rho} |k(x)| \, dx \leq c |u| + \frac{1}{\rho} \int |k(x)| \, dx \leq \varepsilon |u|.
\]

Let now \( h(x) = k(x) \) if \( \varepsilon < |x| < \delta \), \( h(x) = 0 \) otherwise. Then, if \( 4|y| < |x| \), we have \( h(x - y) - h(x) = k(x - y) - k(x) \) if \( (4/3)\varepsilon < |x| < (4/3)\delta \) and \( h(x - y) = h(x) = 0 \) if \( |x| < (4/5)\varepsilon \) or \( |x| > (4/3)\delta \). Consider now the spherical shells between the spheres of radii \((4/5)\varepsilon\), \((4/3)\varepsilon\), \((4/3)\delta\), \((4/3)\delta\) and center at \( x = 0 \) and let \( S_1, S_2, S_3 \) be the portions of these shells contained in \( 4|y| < |x| \). Using the preceding inequality, we get

\[
\int_{|x| > 4|y|} |h(x - y) - h(x)| \, dx \leq \int_{|x| > 4|y|} \left| k(x - y) - k(x) \right| \, dx + \int_{|x| > 4|y|} |k(x)| \, dx + \int_{|x| > 4|y|} |k(x - y) - k(x)| \, dx \leq \varepsilon |u|.
\]

Consequently \( h \) satisfies \((i), (ii)\), and the first inequality in \((iii)\) and thus also

\[
\int_{|x| \leq 2\rho} |h(x - y)| \, dx \leq \varepsilon.
\]

Let now \( \hat{h} \) be the Fourier transform of \( h \). Then, setting \( z = x/2 \) \(|x|^2 \), we have

\[
2\hat{h}(x) = 2 \int e^{2\pi i (x \cdot y)} h(y) \, dy = \int e^{2\pi i (x \cdot y)} [h(y) - h(y - z)] \, dy = \int e^{2\pi i (x \cdot y)} [h(y) - h(y - z)] \, dy + \int e^{2\pi i (x \cdot y)} h(y) \, dy.
\]

where the last integral is extended over \(|y| > 4|z|\), \(|y - z| < 5|z|\). Observing that \( e^{2\pi i (x \cdot y)} \leq c |x| |y| \) and \( e^{2\pi i (x \cdot y)} + 1 \leq c |x| |y - z| \), using the preceding inequality, and \((i), (ii)\), and the first inequality in \((iii)\), which are valid for \( c \) replaced by \( \varepsilon \), we can estimate \( |\hat{h}(x)| \) from the last expression and find that \( |\hat{h}(x)| \leq \varepsilon |u| \). Now Plancherel's identity is valid for Fourier transforms of Hilbert space valued functions. This is readily seen by expressing the elements of the Hilbert space in terms of an orthonormal basis. Consequently, we have

\[
\|h \ast f\|_2 = \|\mathcal{F}h(x - y)f(y)\|_2 = \|\hat{h} \|_2 \leq \varepsilon \|f\|_2 = \varepsilon \|\hat{f}\|_2 = \varepsilon \|f\|_2
\]

for \( f \in L^p_0(H) \) with \( \varepsilon \) independent of \( \varepsilon \) and \( \delta \). Now for \( f \in L^p_0(H) \) we have \( (h \ast f)(x) \rightarrow (A_f)(x) \) for each \( x \) as \( \varepsilon \rightarrow \infty \). Hence \( \|A_f\|_2 \leq \varepsilon \|f\|_2 \) with \( \varepsilon \) independent of \( \varepsilon \), and by Theorem 2 we have, according to the case, \( \|A_f\|_p \leq c_p \|f\|_p \) or \( \|A_f\|_p \leq c_p \|f\|_p \) with \( c_p \) and \( c_p \) independent of \( \varepsilon \). To show that \( A_f \) converges in \( L^p(H) \) as \( \varepsilon \rightarrow 0 \), it is enough to show now that this is the case for say \( f \) continuously differentiable with compact support. Under these circumstances, we have \( |f(x) - f(y)| \leq c |x - y| \) and consequently, by \((i)\),
\[(A,f)(x) = \int_{|x-y|>1} k(x-y)[f(y) - f(x)] dy + \int_{|y|>1} [\int_{|x-y|>1} k(y) dy] f(x) + \int_{|x-y|>1} k(x-y)f(y) dy\]

converges for each \(x\), uniformly on compact sets. Since in addition \(A_f\) is independent of \(\epsilon\) for \(\epsilon < 1\) and \(|x|\) sufficiently large, it follows that \(A_f\) converges in \(L^p(H_2)\) or \(L^p(H_2')\), according to the case. Our theorem is thus established.

The following result is useful to determine the existence of inverses to the operators under consideration.

**Theorem 4.** Let \(h(x)\) be a function whose values are bounded operators from a Hilbert space \(H_1\) to a Hilbert space \(H_2\) such that the functions \(h(px)\) of \(x, 0 < p < \infty\), are equicontinuous with respect to the operator norm in \(1 < |x| < 2\) and \(|h(x)|\) is bounded. Let \(Af, f \in L^p_0(H_1)\) be defined by \((Af)^\sim = h(x)f(x)\) where \((Af)^\sim\) and \(f\) are the Fourier transforms of \(Af\) and \(f\) respectively. Suppose that \(\|Af\|_p \leq c_p\|f\|_p\) for all \(P = (p_1, p_2, \ldots, p_n), 1 < p_i < \infty\). Then if \(A\) has a bounded inverse as an operator from \(L^p(H_1)\) to \(L^p(H_2)\), it also has a bounded inverse as an operator from \(L^p(H_2)\) to \(L^p(H_2)\), for all \(P = (p_1, \ldots, p_n), 1 < p_i < \infty\).

**Proof:** Let \(B\) be a linear operator from \(L^p_0(H_1)\) to \(M(H_1)\). Let \(\|B\|_p = \sup\|BF\|_p/\|f\|_p\) and \(\|B\|_2 = \sup\|BF\|_2/\|f\|_2\). Then if \(P = (p_1, \ldots, p_n)\) and \(P = (p_1, \ldots, p_n), 1/p_i = s/p_i^{(1)} + (1 - s)/2, 0 < s < 1\), we have \(\|B\|_p \leq \|B\|_p^{(1)}\|B\|_2^{s-1}\). This is well known (see ref. 1, Theorem 0) if \(H_1\) is one-dimensional. In the general case, consider the operator acting on bounded measurable functions \(g(x)\) with compact support defined by \(B(g) = (Bfg, \mu)\), where \(u(x)\) and \(\nu(x)\) have values in \(H_1, |u(x)| = |\nu(x)| = 1\) and the parenthesis denotes inner product. Then we have \(\|B\|_p \leq \|B\|_p^{(1)}\|B\|_2^{1-s} \leq \|B\|_p^{(1)}\|B\|_2^{1-s}\), that is, \(\|Bg\|_p \leq \|B\|_p^{(1)}\|B\|_2^{1-s}\). Setting \(g = f, \mu = |Bf| = Bf\), we get \(\|B\|_p \leq \|B\|_p^{(1)}\|B\|_2^{1-s}\) as asserted. Applying this result to \(B^m\), we have \(\|B^m\|_p^{1/m} \leq \|B^m\|_p^{1/m}\|B\|_2^{1-s}\) and letting \(m\) tend to infinity to obtain \(\|B\|_p \leq \|B\|_p^{(1)}\|B\|_2^{1-s}\) where \(|Bf|, |B|, |B|_p, |B|_p, |B|_2\) are the spectral norms of \(B\) as an operator in the corresponding spaces.

Let now \(k(x)\) be a function whose values are bounded operators in \(H_1\). Suppose that \(k\) has infinitely many continuous derivatives in \(x \neq 0\), and derivatives of order \(m\) are \(O(|x|^{-m})\) in norm. Let \(B\) be defined by \((Bf)^\sim = k(x)f(x)\). Then \(\|B\|_p\) is finite for all \(P = (p_1, \ldots, p_n), 1 < p_i < \infty\). This is a weak vectorial version of a theorem of Mihlin which can be derived from Theorem 2 as it is done in reference 5, Theorem 2.5. Let now \(I\) be the identity operator and \(\lambda\) a number such that \(B - \lambda I\) has a bounded inverse in \(L^p(H_1)\). Then, \(k(x) - \lambda I\) has a bounded inverse \((k - \lambda I)^{-1}(x)\) for each \(x\) and \(|k - \lambda I|^{-1}(x)\) is bounded. Furthermore \((k - \lambda I)^{-1}(x)\) has infinitely many continuous derivatives in \(x \neq 0\), and differentiating successively the identity \((k - \lambda I)(k - \lambda I)^{-1} = I\) an inductive argument shows that derivatives of order \(m\) of \((k - \lambda I)^{-1}(x)\) are \(O(|x|^{-m})\) in norm. Consequently, \(\|B - \lambda I\|_p\) is finite for all \(P, 1 < p_i < \infty\), and \(B - \lambda I\) has a continuous inverse as an operator in \(L^p(H_1)\). Consequently, the spectrum of \(B\) as an operator in \(L^p(H_1)\) is contained in its spectrum as an operator in \(L^p(H_1)\). In particular, we have \(\|B\|_p \leq |B|_2\).

Suppose now that in the hypothesis of the theorem we have \(H_1 = H_2\). The equicontinuity of \(h(px)\) in \(1 < |x| < 2\) implies that \(h(px)\) is uniformly continuous in the cartesian product of the line \(-\infty < t < \infty\) and the spherical shell \(1 < |x| < 2\). Then given \(\epsilon > 0\) there exists an infinitely differentiable function \(l(x, t)\) which is bounded in norm and has derivatives of all orders bounded in norm such that
\[ |l(x, t) - h(e^l(x))| < \epsilon \text{ in } -\infty < t < \infty, \quad 1 < |x| < 2. \] Setting \( k(x) = l(x, \log |x|) \), we find that \( |k(x) - h(x)| < \epsilon \). Furthermore, \( k \) has continuous derivatives of all orders in \( x \neq 0 \) and, as an elementary calculation shows, derivatives of order \( m \) of \( k(x) \) are \( O(\log |x|^{-m}) \) in norm. Consequently, if \( (Bf)^+ = k(x)f(x) \), we have

\[ \|B\| \leq \|A\| + \|A - B\| \leq \|A\| + \epsilon \quad \text{for all } \|P\| = (p_1, \ldots, p_n), \quad 1 < p_i < \infty. \]

Let now \( P = (p_1, p_2, \ldots, p_n), \quad 1 < p_1 < \infty \) be given and let \( s \) and \( P_1 = (p_1^{(1)}, \ldots, p_n^{(1)}) \), \( 0 < s < 1, \quad 1 < p_i^{(1)} < \infty \) be related to \( P \) as above. Then,

\[ \|A - B\|_p \leq \|A - B\|_p, \quad \|A - B\|_2 \leq \sum \left|\left(A\right)_{ij} + \left(B\right)_{ij}\right|^{1-s} \leq \left(\|A\|_p + \|A - B\|_p\right)^{1-s} \leq \left(\|A\|_2 + \epsilon + \|\left(A\right)_{ij} + \left(B\right)_{ij}\right|^{1-s}\right), \]

and since \( \epsilon \) is arbitrary, it follows that \( \|A\|_p \leq \|A\|_s \). Finally, let \( A \) be as postulated in our theorem and let \( \bar{A} \) be the operator from \( L_0^q(\mathcal{H}_2) \) to \( M(\mathcal{H}_1) \) defined by \( (\bar{A}f)^+ = k^*(x)f(x) \). Then \( \bar{A} \) also satisfies the hypothesis of our theorem. To see this, we merely have to verify that \( \|\bar{A}\|_p \leq \|\bar{A}\|_q \) for all \( P = (p_1, \ldots, p_n), \quad 1 < p_i < \infty \). Let \( Q = (q_1, \ldots, q_n), \quad q_i = p_i/(p_i - 1) \) and let \( f \) have values in \( \mathcal{H}_1 \) and be integrable on compact sets. Then

\[ \sup_{g} |\mathcal{J}(f, g)|dx = \sup_{g} |\mathcal{J}(f)|dx = \|f\|_p \]

where the supremum is taken over all \( g \in L_0^q(\mathcal{H}_1) \) such that \( \|g\|_q \leq 1 \) (See ref. 1, Theorem 2.1). On the other hand, by Plancherel's identity we have

\[ \mathcal{J}(\bar{A}f, g)dx = \mathcal{J}(f, Ag)dx \]

and

\[ \|\bar{A}f\|_p = \sup_{g} |\mathcal{J}(\bar{A}f, g)|dx = \sup_{g} |\mathcal{J}(f, Ag)|dx \leq \|f\|_p \|A\|_q, \quad \|g\|_q \leq 1. \]

Thus \( \bar{A} \) satisfies the hypothesis of the theorem and \( \bar{A}A \) does also. Now, if \( A \) has a bounded inverse as an operator from \( L^p(\mathcal{H}_1) \) to \( L^q(\mathcal{H}_2) \), then \( k^*(x)k(x) \) has an inverse for each \( x \) whose norm is a bounded function of \( x \). Since \( k^*(x)k(x) \) is selfadjoint and positive, there exist two positive numbers \( \rho \) and \( r \) such that \( \rho I \leq k^*(x)k(x) \leq rI \). Consequently,

\[ \frac{\rho - r}{2} I \geq k^*k - \frac{\rho + r}{2} I \geq \frac{r - \rho}{2} I \]

and therefore,

\[ \left[ \bar{A}A - \frac{\rho + r}{2} I \right] = \left[ A A - \frac{\rho + r}{2} I \right] \leq \frac{\rho - r}{2} I \]

Thus the spectrum of \( \bar{A}A \) as an operator in \( L^p(\mathcal{H}_1) \) is contained in the a circle with center at \( (\rho + r)/2 \) and radius \( (\rho - r)/2 \) and hence does not contain the origin. Consequently \( \bar{A}A \) has a bounded inverse in \( L^p(\mathcal{H}_1) \) and \( (\bar{A}A)^{-1} \bar{A} \) is a left bounded inverse of \( A \) as an operator from \( L^q(\mathcal{H}_1) \) to \( L^q(\mathcal{H}_2) \).

Remark: The statement of the theorem remains valid if \( P \) is replaced by \( p, \quad 1 < p < \infty \), throughout.

Applications.—(1) Let \( k(x) \) be a numerical function in \( E^* \) which is homogeneous
of degree \(-n\), continuously differentiable in \(x \neq 0\) and of mean value zero on \(|x| = 1\). Then Theorem 3 applies to the integral \(\int k(x - y)f(y)dy = A_f\) and yields well-known results as well as new estimates for the \(L^p\)-norms of \(A_f\).

(2) Let \(\varphi(x)\) be a numerical function in \(E^n\) satisfying the following conditions:

\[
\begin{align*}
(i) & \quad |\varphi(x)| \leq c(1 + |x|)^{-n - \alpha}, \quad \alpha > 0; \\
(ii) & \quad \int \varphi(x)dx = 0; \\
(iii) & \quad \int |\varphi(x - y) - \varphi(x)| dx \leq c|y|^{\alpha}.
\end{align*}
\]

Then \(t^{-n-\frac{1}{2}}\varphi(x/t)\) is a square integrable function of \(t\) for each \(x, x \neq 0\). Let \(k(x)\) be the operator acting on the space of complex numbers and transforming the number \(a\) into the function \(t^{-n-\frac{1}{2}}\varphi(x/t)a\) of \(t \in L^1(0, \infty)\). As we will see below, \(k\) satisfies the assumptions of Theorem 3 and consequently, if \(f\) is bounded and has compact support and

\[
g = \left[ \int_0^\infty dt \left( \int t^{-n-\frac{1}{2}}\varphi\left(\frac{x - y}{t}\right)f(y)dy \right)^2 \right]^{1/2},
\]

we have

\[
\|g\|_P \leq \lim_{s \to 0} \left\| \left[ \int_0^\infty dt \left( \int t^{-n-\frac{1}{2}}\varphi\left(\frac{x - y}{t}\right)f(y)dy \right)^2 \right]^{1/2} \right\|_P \leq c_P\|f\|_P.
\]

\(P = (p_1, p_2, \ldots, p_n), 1 < p_i < \infty\). By a passage to the limit, we obtain the same inequality for all \(f \in L^P\).

Setting

\[
\varphi(x) = \frac{\partial}{\partial t} \left( \frac{t}{|x|^2 + t^2} \right)_{t=1}
\]

or

\[
\varphi(x) = \frac{1}{\partial x_t} \left( \frac{1}{x^2 + 1} \right)^{(n+1)/2}
\]

we obtain the Littlewood Paley functions in \(E^n\).

Setting \(\varphi(x) = F(x)\) if \(|x| < 1\) and \(\varphi(x) = 0\) if \(|x| > 1\), where \(F(x)\) is a homogeneous function of degree zero which is continuously differentiable in \(x \neq 0\) and has mean value zero on \(x = 1\), the function \(g\) becomes the Marcinkiewicz integral in \(E^n\).

To see that \(\varphi(x)\) satisfies the assumptions of Theorem 3, we first observe that, on account of (ii) we have

\[
\int_{|x| \leq r} \varphi(x)dx = -\int_{|x| > r} \varphi(x)dx,
\]

and by (i) this implies that

\[
\int_{|x| \leq r} \varphi(x)dx \leq c \frac{r^n}{(1 + r)^{n+\alpha}}.
\]

Consequently,

\[
\int_0^\infty \int_{|x| \leq r} t^{-n-\frac{1}{2}}\varphi(x/t)dxdt \leq c,
\]
and this clearly implies that condition (i) of Theorem 3 is satisfied.

On the other hand, from (i) it follows readily that \(|k(x)| \leq \varepsilon |x|^{-\alpha}\), which implies condition (ii) of Theorem 3.

Finally let \(\beta = \alpha/2\); then

\[
\int |k(x - y) - k(x)| \, dx = \int dx \left[ \int_0^1 t^{-\alpha - 1} \phi \left( \frac{x - y}{t} \right) \right. \\
\left. - \phi \left( \frac{x}{t} \right) \right] \left[ \int t^{-\alpha - 1} |x|^{\alpha} \phi \left( \frac{x - y}{t} \right) \right. \\
\left. - \phi \left( \frac{x}{t} \right) \right]^\frac{1}{\alpha} \int |x|^{-\alpha - \beta} \, dx < \varepsilon^\frac{1}{\alpha}.
\]

On account of (i), we have \(|\phi(x - y) - \phi(x)| \leq \varepsilon |x|^{-\beta} |y|^{\alpha} \) for \(|x| > 4 \, |y|\) and since by (i) \(\phi\) is integrable, the right-hand side of (iii) can be replaced by \(\varepsilon |y|^{\alpha}(1 + |y|)\). Thus, the last expression is dominated by

\[
\varepsilon |y|^{-\beta} \int_0^\infty t^{-\alpha - \beta - 1} \, dt \int |x - y|^{-\alpha} \phi \left( \frac{x - y}{t} \right) \left[ \int_0^1 t^{-\alpha - 1} \frac{(|y|/t)^{\alpha}}{1 + (|y|/t)^{\alpha}} \, dt \right]^\frac{1}{\alpha} \leq \varepsilon.
\]

Suppose now that the function \(\phi\) above has values in an \(m\)-dimensional Hilbert space \(H\). Let \(U\) denote a rotation of \(E^n\) about the origin and let \(\pi(U)\) denote an \(m\)-dimensional representation of this rotation group. Suppose that \(\phi\) has the property that \(\phi(Ux) = \pi(U)\phi(x)\). Then if \(\psi\) denotes the Fourier transform of \(\phi\), we also have \(\psi(Ux) = \pi(U)\psi(x)\). Let now

\[
Af = \lim_{\varepsilon \to 0} \int k(x - y)f(y)dy = \int t^{-\alpha - 1/2} \phi \left( \frac{x - y}{t} \right)f(y)dy.
\]

Taking Fourier transforms with respect to \(x\) and using Plancherel's identity, we find that

\[
||Af||^2_{\ell^2} = \int \left| \psi(xt) \right|^{\alpha - 1/2} f(x) \, dx \, dt = \int \left| f(x) \right|^{\alpha} \, dx \int_0^\infty \left| \psi(xt) \right|^{\alpha - 1/2} dt.
\]

But the inner integral in the last expression is independent of \(x\). In fact, if \(y = \rho Ux\), we have

\[
\int_0^\infty \left| \psi(\rho x) \right|^{\alpha - 1/2} \, dt = \int_0^\infty \left| \psi(U\rho x) \right|^{\alpha - 1/2} \, dt = \int_0^\infty \left| \pi(U)\psi(\rho x) \right|^{\alpha - 1/2} \, dt = \int_0^\infty \left| \psi(\rho x) \right|^{\alpha - 1/2} \, dt.
\]

Thus, except for a constant factor, \(A\), as an operator on \(L^2\) with values in \(L^2(H)\), is an isometry. Consequently, \(A^*\)

\[
A^*g = \lim_{\varepsilon \to 0} \int k(x - y)g(x)dx, \quad g(x) = g(x, t)
\]

is, except for a constant factor, a left inverse of \(A\). But Theorem 3 also applies to \(A^*\). Hence,

\[
||f||_P = ||A^*Af||_P \leq c_P ||Af||_P, \quad P = (p_1, \ldots, p_m), \quad 1 < p_i < \infty.
\]
To obtain the well known results about the "area" function of Littlewood Paley and its generalizations, one considers the operator valued function \( k(x) \) which transforms the complex number \( a \) into the function \( t^{-n-1/2}\varphi(x/t - z) \) of \( t \) and \( z \), \( 0 < t < \infty, \|z\| \leq R \), where \( \varphi \) is the function described in (2). Then as in (2), we find that if

\[
g(x) = \left[ \int_0^\infty dt \int dz \left| \int t^{-n-1/2} \varphi \left( \frac{x - y}{t} - z \right) f(y)dy \right|^2 \right]^{1/2} < R
\]

then \( \|g\|_P \leq c_P \|f\|_P \), \( P = (p_1, \ldots, p_n) \), \( 1 < p_i < \infty \).

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† See the papers quoted below.


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ENTSCHIEDUNGSPROBLEM REDUCED TO THE \( \forall \exists \forall \) CASE

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Communicated by Claude E. Shannon, December 4, 1961

1. The Main Theorem and Its Immediate Consequences.—By an \( \forall \exists \forall \) formula, we mean a formula of the predicate calculus of the form \( \forall x \exists u \forall y \exists z \phi(x, u, y, z) \), where \( \phi(x, u, y, z) \) is quantifier-free and contains neither the equality sign nor function symbols. By a restricted \( \forall \exists \forall \) formula is meant an \( \forall \exists \forall \) formula which contains only dyadic predicate letters and, for each dyadic \( \phi_i \), only basic components of the forms \( G_{i}, G_{iy}, G_{iyx}, G_{iyy} \). By the decision problem of the (restricted) \( \forall \exists \forall \) case, we mean the problem of finding a general algorithm to decide, for each given (restricted) \( \forall \exists \forall \) formula, whether it is satisfiable (as always, in some nonempty domain).

Main Theorem (this contains two parts). \( MT_a \): The decision problem of the restricted \( \forall \exists \forall \) case is unsolvable. \( MT_b \): The class of restricted \( \forall \exists \forall \) formulas is a reduction type; in other words, there is a general algorithm by which each formula \( F \) of the predicate calculus, including the equality sign and function symbols, can be transformed into a restricted \( \forall \exists \forall \) formula \( F^* \) such that \( F \) is satisfiable if and only if \( F^* \) is.

As it happens, the proof of \( MT_a \) and \( MT_b \) can be modified to give slightly stronger results which deal with even narrower classes. In the specification of the class of