$$0 < \psi_i < \infty, \qquad \sum_i \psi_i^{-1} < \infty$$

but otherwise arbitrary. Then the matrix $(\psi_i\psi_{ij})$ is a conservative Q-matrix. Associated with it are the two systems of Kolmogorov differential equations. Let the minimal solution to both systems be (ρ_{ij}); this corresponds to the $(\bar{p}_{ij})$ above and the corresponding minimal chain is a continuous parameter dilation of the given discrete parameter chain. Now the crucial, but simple, formula is

$$\int_0^\infty \rho_{ij}(t)dt = \sum_{n=0}^{\infty} \varphi_{ij}^{(n)}\psi_j^{-1},$$

which shows that the MDH boundary for the discrete parameter chain is the same as that for the continuous parameter dilation.

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† Formal analogues of Doob's results were studied in reference 4.

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A SOLVABILITY CRITERION FOR FINITE GROUPS AND SOME CONSEQUENCES

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The object of this note is to discuss the following theorem, a proof of which will appear in the *Pacific Journal of Mathematics.*

THEOREM 1. *All finite groups of odd order are solvable.*

We first mention two consequences of Theorem 1.

THEOREM 2. *If $\mathfrak{H}$ is a normal subgroup of the group $\mathfrak{G}$ and the order and index of $\mathfrak{H}$ are coprime, then there is a complement to $\mathfrak{H}$ in $\mathfrak{G}$ and any two such complements are conjugate in $\mathfrak{G}$.*

It is well known (see ref. 3, p. 162) that Theorem 2 is a consequence of Theorem 1.

THEOREM 3. *Let $\mathfrak{G}$ be a group of order g. If $g \neq 1$ and g is not a prime, then $\mathfrak{G}$ contains a proper subgroup $\mathfrak{H}$ of order h such that $h^3 \geq g$.*

Proof: The result is trivial for solvable groups. If g is even, the theorem was proved in ref. 1. Hence, Theorem 1 implies the validity of Theorem 3 in all cases.

In the proof of Theorem 1, results of P. Hall are used frequently. H. Wielandt's important idea of weak closure is used in tandem with Theorem B of Hall and Higman at several points. Results of N. Blackburn and B. Huppert are helpful. The results of R. Brauer and M. Suzuki on exceptional characters form the base of certain generalizations needed here.

Proceeding by way of contradiction, a counterexample $\mathfrak{G}$ of least order is a simple group of odd composite order all of whose proper subgroups are solvable.

If $\mathfrak{P}$ is a Sylow p-subgroup of the p-solvable group $\mathfrak{S}$ and $\mathfrak{A}$ is a maximal normal abelian subgroup of $\mathfrak{P}$, an easy lemma shows that the set of p'-subgroups of $\mathfrak{S}$ normalized by $\mathfrak{A}$ is a lattice under intersection and group-theoretic join whose maximal element is $\mathbf{O}_{p'}(\mathfrak{S})$, the largest normal p'-subgroup of $\mathfrak{S}$. This characterization of $\mathbf{O}_{p'}(\mathfrak{S})$ is used frequently in the study of the proper subgroups of $\mathfrak{G}$.

It is shown that if $\mathfrak{A}$ is a maximal normal abelian subgroup of the S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$ and $\mathfrak{A}$ is not generated by two elements, then for each prime $q \neq p$, the q-subgroups of $\mathfrak{G}$ which are maximal with respect to the property of being normalized by $\mathfrak{A}$ are permuted transitively under conjugation by the centralizer of $\mathfrak{A}$.

Another result deals with the discovery of triples $(\mathfrak{Q}, \mathfrak{B}, \mathfrak{C})$, where $\mathfrak{B}$ and $\mathfrak{C}$ are subgroups of the p-subgroup $\mathfrak{Q}$ of $\mathfrak{G}$, with the property that whenever $\mathfrak{Q}$ is an S_p-subgroup of the proper subgroup $\mathfrak{H}$ of $\mathfrak{G}$, then $\mathbf{O}_{p'}(\mathfrak{H})$, $\mathfrak{H} \cap \mathbf{N}(\mathfrak{B})$ and $\mathfrak{H} \cap \mathbf{N}(\mathfrak{C})$ generate $\mathfrak{H}$, $\mathbf{N}$ denoting normalizer.

Denote by π the set of primes p such that $\mathfrak{G}$ contains an elementary subgroup of order p^3, and say that $p \sim q$ if and only if $\mathfrak{G}$ contains a proper subgroup $\mathfrak{K}$ which contains elementary subgroups of orders p^3 and q^3. Using the above results, together with properties of solvable groups, $\sim$ is shown to be an equivalence relation on π, and if σ is an equivalence class, then $\mathfrak{G}$ contains a Hall σ-subgroup $\mathfrak{H}$, $\mathbf{N}(\mathfrak{H})$ is a maximal subgroup of $\mathfrak{G}$, and $\mathfrak{H} \cap \mathfrak{H}^G$ is of square free order for each G in $\mathfrak{G} - \mathbf{N}(\mathfrak{H})$. Furthermore, elements of $\mathfrak{H}$ are $\mathfrak{G}$-conjugate if and only if they are $\mathbf{N}(\mathfrak{H})$-conjugate.

The preceding results lead to rather precise information concerning the structure and embedding in $\mathfrak{G}$ of the maximal subgroups of $\mathfrak{G}$. If $\mathfrak{L}$ is a maximal subgroup of $\mathfrak{G}$, then $\mathfrak{L}$ contains a "large" subset $\hat{\mathfrak{L}}$ with a variety of properties. If $\mathcal{C}(\hat{\mathfrak{L}})$ denotes the set of class functions of $\mathfrak{L}$ vanishing outside $\hat{\mathfrak{L}}$ and $\mathcal{C}_o(\hat{\mathfrak{L}})$ the subset of $\mathcal{C}(\hat{\mathfrak{L}})$ vanishing also at the identity, these properties may be used to establish a mapping τ from $\mathcal{C}_o(\mathfrak{L})$ into the class functions of $\mathfrak{G}$. The mapping τ is in addition a linear isometry mapping generalized characters of $\mathfrak{L}$ into generalized characters of $\mathfrak{G}$, and $\alpha^{\tau}(L) = \alpha(L)$ for all L in $\hat{\mathfrak{L}}$ and α in $\mathcal{C}_o(\hat{\mathfrak{L}})$. Thus, τ is the bridge between local and global information. Criteria are then given which guarantee that τ can be extended to certain subsets of $\mathcal{C}(\hat{\mathfrak{L}})$. If such an extension is available, delicate information regarding the values of some of the irreducible characters of $\mathfrak{G}$ can be obtained. If no such extension is available, the structure of $\mathfrak{L}$ is severely limited.

An elaborate analysis shows that if $\mathfrak{L}$ is a maximal subgroup of $\mathfrak{G}$ and $\mathfrak{L}$ contains no cyclic subgroup which is self-normalizing in $\mathfrak{G}$, then $\mathfrak{L}$ is a Frobenius group. This result, in conjunction with reference 2, shows that $\mathfrak{G}$ contains a self-normalizing cyclic subgroup $\mathfrak{W}$. Furthermore, $\mathfrak{W} = \mathfrak{W}_1 \times \mathfrak{W}_2$, and if $\hat{\mathfrak{W}}$ is the set of elements of $\mathfrak{W}$ which are in neither $\mathfrak{W}_1$ nor $\mathfrak{W}_2$, then $\mathfrak{W}$ is the normalizer of every nonempty subset of $\hat{\mathfrak{W}}$. There are two nonconjugate maximal subgroups $\mathfrak{S}$, $\mathfrak{T}$ with $\mathfrak{S} \cap \mathfrak{T} = \mathfrak{W}$ and every maximal subgroup of $\mathfrak{G}$ which is conjugate neither to $\mathfrak{S}$ nor to $\mathfrak{T}$ is a Frobenius group.

Exceedingly precise information is obtained about the irreducible characters of $\mathfrak{G}$ which do not vanish on $\hat{\mathfrak{W}}$. So precise is this information that the accumulated results lead to an explicit determination of $\mathfrak{S}$ and $\mathfrak{T}$ in terms of generators and relations. $\mathfrak{W}$ has order pq, where p and q are distinct primes with $p > q$. $\mathfrak{S}$ is isomorphic to the group of all mappings $\varphi_{a,b,\sigma} : x \rightarrow ax^{\sigma} + b$, where $x, a, b \in F(p^q)$, the field of p^q elements, a has norm 1, that is, $a^{1+p+\cdots+p^{q-1}} = N(a) = 1$, and σ ranges over the automorphisms of $F(p^q)$. This fact, together with the structure of $\mathfrak{T}$ and the embeddings of $\mathfrak{S}$, $\mathfrak{T}$ in $\mathfrak{G}$, leads to information about $F(p^q)$. Specifically, it is shown that if a is in $F(p^q)$ and $N(a) = N(2 - a) = 1$, then $N(2a - 1) = 1$. It is then an easy matter to find an element a of $F(p^q)$ such that $N(a) = N(2 - a) = 1 \neq N(2a - 1)$, this being the desired contradiction.

The assumption that $\mathfrak{G}$ has odd order is used in several ways. Burnside's result that no nonprincipal irreducible character of $\mathfrak{G}$ is real-valued is used continually. The character computations yield various inequalities in which an exponential is less than a polynomial, the variable being a prime p dividing the order of $\mathfrak{G}$. This yields that p is small, and it is then often possible to show that $p < 3$, thus leading to a contradiction. The proof could be shortened considerably if it could be shown directly that 3 does not divide the order of $\mathfrak{G}$.

The proof would be simplified considerably if it is true that nonabelian simple groups never contain self-normalizing cyclic subgroups. The validity of the conjecture that $(p^q - 1)/(p - 1)$ never divides $(q^p - 1)/(q - 1)$ if p,q are distinct primes would also simplify the proof, rendering unnecessary the detailed use of generators and relations. If it is true that nonidentity Sylow subgroups of simple groups always contain nonidentity abelian weakly closed subgroups, short proofs of the necessary group-theoretic lemmas could be given.

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ON INCREASING MARKOV PROCESSES

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1. *Introduction.*—A Markov process on the line is said to be increasing if its paths are almost always nondecreasing. It is said to satisfy condition (A) if (1) its Green's operator $G_{\alpha}(a,db) = \int_0^{\infty} e^{-\alpha t} P(t,a,db)dt$, $P(t,a,db)$ being the transition probabilities of the process, maps continuous functions vanishing at $+\infty$ into