BOUNDS OF ANALYTIC FUNCTIONS OF TWO COMPLEX VARIABLES IN DOMAINS WITH THE BERGMAN-SILOV BOUNDARY

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Let $\mathfrak{B}$ be a bounded domain in the space $\mathfrak{C}$ of two complex variables $z_1, z_2, z_3 = x_3 + iy_3, \kappa = 1, 2.$ ($\kappa$ assumes everywhere the values 1 and 2.) Its boundary $\partial \mathfrak{B}$ consists of finitely many segments $\partial^j, j = 1, \ldots, n$, of analytic surfaces. Each $\partial^j$ is given by the parametric representation

$$ z_3 = h_{\lambda_j}(Z, \lambda_j), \quad (1) $$

where $h_{\lambda_j}$ are continuously differentiable functions of $Z = \{Z_k, \lambda_k\}$ in the set $\{Z_k, \lambda_k\} | Z_k | \leq 1, \lambda_k \in (0, 2\pi)$, $(1a)$ $(1a = \text{hypothesis} (1a))$ and

$$ (h_{\lambda_1}(Z', \lambda'), h_{\lambda_2}(Z', \lambda')) \neq (h_{\lambda_1}(Z, \lambda_1), h_{\lambda_2}(Z, \lambda_i)) \quad (2) $$

for $(Z', \lambda') \neq (Z, \lambda), |Z'\rangle, |Z\rangle < 1 \text{ (1b).}$ For every point $(z_1, z_2, z_3), z_3^0 = h_{\lambda_j}(Z, \lambda_j), |Z| < 1$, and for every sufficiently small neighborhood $\mathfrak{U}$ of this point, there exists $\alpha > 0$ such that the set of points of $\lambda^j$ which belong to $\mathfrak{U}$ consists of many points (1) for which $|Z_3 - z_3| < \alpha$ and $|\lambda_j - \lambda_i| < \alpha$ hold, $(1c)$. For a fixed $k$ and $\lambda_k$, the corresponding set of points (1) is called a lamina of $\partial^j$ and is designated by $\partial_{\lambda_j}(\lambda_k)$. The set $\partial_{\lambda}^j$ of points (1) corresponding to the values $|Z_j| = 1, k = 1, \ldots, n$, constitutes the so-called Bergman-Silov boundary surface of $\mathfrak{B}$ on which the maximum principle is valid for functions $f(z_1, z_2)$ holomorphic in $\mathfrak{B}$ and continuous in $\mathfrak{B}$ (see ref. 1).

Let $\partial \mathfrak{B}^j$ designate an analytic surface of the form $z_3 = g_3(\xi), \xi \in \partial \mathfrak{B}$, where $\partial \mathfrak{B}$ is a domain in the $\xi$-plane and $g_3$ are functions regular in $\xi$ and continuous in $\partial \mathfrak{B}$. We assume that $\partial \mathfrak{B}^j$ has common points with $\partial \mathfrak{B}$ and its whole boundary lies in $\mathfrak{C} \setminus \mathfrak{B}$. Concerning the intersection of $\partial \mathfrak{B}^j$ with $\partial \mathfrak{B}$, we list the following properties to be assumed as indicated in the various Theorems 1–4). The intersection $\partial \mathfrak{B}^j \cap \partial \mathfrak{B}$ has the following representation $\partial \mathfrak{B}^j = \{z_3 | z_3^j = g_j(\xi), \xi \in \partial \mathfrak{B} \cap \partial \mathfrak{B} \}, (2a)$. The boundary curve $\partial \mathfrak{B}^j$ is simultaneously the intersection $\partial \mathfrak{B}^j$ with $\partial^j$, $(2b)$. $g_j^j = \{z_3, z_3 = g_j(e^{i\varphi}), \varphi \in (0, 2\pi)\}$ can be divided into $J$ parts: $g_j^j = \{z_3, z_3 = g_j(e^{i\varphi}), \varphi \in (\varphi_{j_1}, \varphi_{j_{1+1}}), \varphi \in (\varphi_j, \varphi_{j+1})\}, j = 1, \ldots, J, \varphi_j < \varphi < \varphi_{j+1} = \varphi_j + 2\pi$, so that $g_j^j \subset \partial \mathfrak{B}^j$, $1 \leq k_j \leq n$, $k_{j_1} \neq k_{j_2}$ if $j_1 \neq j_2$, and only the points $(g_j(e^{i\varphi}), g_j(e^{i\varphi}))$ belong to $\partial \mathfrak{B}^j$, $(2c)$. Every point of $g_j^j$ lies in a certain lamina, say $\partial_{\lambda_j^j}(\lambda_j^j)$. Hence, by (2)

functions $\lambda_j^j = \lambda_j^j(\varphi)$, $Z_{j}^j = Z_{j}^j(\varphi), \varphi \in (\varphi_{j_1}, \varphi_{j_{1+1}})$, exist such that $g_j^j = \{z_3, z_3 = h_{\lambda_j^j}(Z, \lambda_j^j(\varphi), Z_{j}^j(\varphi), \varphi \in (\varphi_{j_1}, \varphi_{j_{1+1}})\}$. We assume that $\lambda_j^j(\varphi) \text{ and } Z_{j}^j(\varphi)$ are continuous in $(\varphi_{j_1}, \varphi_{j_{1+1}})$, $(2d)$. $\lambda_j^j(\varphi)$ are monotone in $(\varphi_j, \varphi_{j+1})$ and $|\lambda_j^j(\varphi)| < 1/Q$, $Q > 0, \varphi \in (\varphi_j, \varphi_{j+1}), j = 1, \ldots, J$, $(2e)$. The expressions $1 - |Z_{j}^j(\varphi)|$ go to zero not faster than some positive power of $\varphi - \varphi_j$ or $\varphi - \varphi_{j+1} \text{if } \varphi \rightarrow \varphi_j \text{ or } \varphi \rightarrow \varphi_{j+1}$, respectively, $(2f)$.

Concerning the function $f(z_1, z_2)$ of two complex variables $z_1, z_2$, we list the following hypotheses to be assumed in the various theorems (see below). $f(z_1, z_2)$ is regular in
the set $\mathfrak{B}_1 = \mathfrak{B} \setminus \mathfrak{B}^2$ and continuous in $\mathfrak{B}_2 = \mathfrak{B}_1 \cup \mathfrak{B}^1 \cap \mathfrak{B}_1$, (3a). $f(z_1, z_2) \neq 0$ in $\mathfrak{B}_2$, (3b). Let $n_\epsilon(Re^\theta, \lambda_\epsilon)$ denote the number of $Re^\theta$-points of $f(z_1, z_2)$ in the lamina $3_2^1(\lambda_\epsilon)$. Let $p_j(\lambda_\epsilon) = \sup_{0 < R < \infty} \frac{1}{2\pi} \int_0^\pi \frac{1}{2\pi} \int_0^{2\pi} n_\epsilon(Re^\theta, \lambda_\epsilon) d\theta d(\rho^2)$. We assume that \[
\frac{1}{f_j} \sum_{j=1}^J \left( \frac{1}{2\pi} \int_{\alpha_j} p_j^2(\lambda_\epsilon) d\lambda_\epsilon \right)^{1/2} \leq P, \quad P > 0, \quad \text{where } \langle \alpha_j, \beta_j \rangle = \lambda_\epsilon(\varphi_j, \varphi_{j+1}), \quad j = 1, \ldots, J, (3c). \]
$p_j(\lambda_\epsilon) \leq P, \lambda_\epsilon \in \langle \alpha_j, \beta_j \rangle$, (3c). Let $l^1 = \{z_1^0, z_2^0 = h_{\lambda_\epsilon}(0, \lambda_\epsilon), \lambda_\epsilon \in \langle \alpha_j, \beta_j \rangle \}$ and $l^1 = \cup l^1$. We set $l = \min_{\rho} |f(z_1, z_2)|$, $L = \max_{\rho} |f(z_1, z_2)|$, $l = \min (1, l), L = \max (1, L)$. We assume in the following that $\mathfrak{B}$ satisfies the conditions 1a-1c.

**Theorem 1.** Let $\mathfrak{B}_0^2$ satisfy conditions 2a–2f. For every $\epsilon > 0$, there exists $r_0, 0 < r_0 < 1$, such that at every point of $\mathfrak{B}^2$, say at $z_0^0 = g_\epsilon(z_0)$, and for every function $f(z_1, z_2)$ which has the properties 3a–3c, the inequality \[
1 + |f(z_1, z_2)| \leq \left( e^{-2\pi P - 1/2 - \epsilon} \left( \frac{1 - r_0}{1 + r_0} \right)^{2JP} \right)^{1/\epsilon} \leq |f(z_1^0, z_2^0)| \leq \left( e^{2\pi P + 1/2 + \epsilon} \right)^{1/\epsilon} \left( \frac{1 + r_0}{1 - r_0} \right)^{2JP} \] holds.

**Theorem 2.** Let $\mathfrak{B}_0^2$ satisfy 2a–2d, 2f. For every $\epsilon > 0$, there exists $r_0, 0 < r_0 < 1$, such that at every point of $\mathfrak{B}^2$, say at $z_0^0 = g_\epsilon(z_0)$, and for every function $f(z_1, z_2)$ which has the properties 3a, 3b, 3c, the inequality \[
1 + |f(z_1, z_2)| \leq \left( e^{-2\pi P - 1/2 - \epsilon} \left( \frac{1 - r_0}{1 + r_0} \right)^{2P} \right)^{1/\epsilon} \leq |f(z_1^0, z_2^0)| \leq \left( e^{2\pi P + 1/2 + \epsilon} \right)^{1/\epsilon} \left( \frac{1 + r_0}{1 - r_0} \right)^{2P} \] holds.

**Theorem 3.** Let $\mathfrak{B}_0^3$ satisfy 2a–2d, 2f. For every $\epsilon > 0$, there exists $r_0, 0 < r_0 < 1$, such that at every point of $\mathfrak{B}^3$, say at $z_0^0 = g_\epsilon(z_0)$, and for every function $f(z_1, z_2)$ which has the properties 3a, 3b, 3c, the inequality \[
\frac{1 - |z_0^0|}{1 + |z_0^0|} \left( \frac{1 - r_0}{1 + r_0} \right)^{2P} - \epsilon \leq |f(z_1^0, z_2^0)| \leq \frac{1 + |z_0^0|}{1 - |z_0^0|} \left( \frac{1 + r_0}{1 - r_0} \right)^{2P} + \epsilon \] holds. We set $\mathfrak{B}_0^\delta = \{z_1, z_2 = h_{\lambda_\epsilon}(z_1, \lambda_\epsilon), \lambda_\epsilon \in (0, 2\pi) \}$.\n
**Theorem 4.** Let $\mathfrak{B}_0^3$ satisfy the conditions 2a and 2b and moreover let $g_1^1 \subset \mathfrak{B}_0^\delta$. For every $(z_1^0, z_2^0) \in \mathfrak{B}^3$ and for every function $f(z_1, z_2)$ which has properties 3a, 3b and 3c, the inequality \[
\frac{1}{1 + |z_0^0|} \left( \frac{1 - r_0}{1 + r_0} \right)^{2P} \leq |f(z_1^0, z_2^0)| \leq \frac{1 + |z_0^0|}{1 - |z_0^0|} \left( \frac{1 + r_0}{1 - r_0} \right)^{2P} \] holds.
Note that in (3)–(6) bounds for $|f|$ are expressed in terms of the minimum and maximum of $|f|$ on a one-dimensional boundary manifold $1$. This manifold is independent of $(z_1^0, z_2^0)$ and of $f$.

In the proof of Theorems 1 and 2, we express the value of $f(z_1, z_2)$ at $(z_1^0, z_2^0)$ by using the Poisson formula applied to the harmonic function log $|f(g_1(\zeta), g_2(\zeta))|$. For the proof of Theorem 3, we apply the Cauchy integral formula to the holomorphic function $f(g_1(\zeta), g_2(\zeta))$, and for Theorem 4 the maximum principle.

Similar results are obtained by Bergman $1^6, 7$ and Charzyński $8$. Bergman considered functions which omit in every lamina two values and functions which are univalent on every lamina. Charzyński assumes that the functions

$$f(h_{1k}(Z_1, \lambda_k), h_{2k}(Z_2, \lambda_k))$$

as functions of $Z_k$ belong to a normal family (but he considers only the case in which $g^1 \subset \mathcal{C}_{h_{1k}}$). The first step in the proof of the Theorems 1–3 is similar to the approach used in references 2 and 3. However only the use of completely new additional procedures will yield our result.

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THE EFFECT OF EXOGENOUS RNA AND DNA ON AMINO ACID INCORPORATION BY SUBCELLULAR FRACTIONS PREPARED FROM ERYTHROID TISSUES*

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This paper describes the components of cell-free systems, obtained from the bone marrow of normal rabbits and from the bone marrow and reticulocytes of acetylphenylhydrazine (APH) treated rabbits, capable of incorporating amino acids into trichloracetic acid-insoluble products. The effects of exogenous RNA and DNA in these systems are described.

Materials and Methods.—Bone marrow subcellular fractions: Bone marrow, obtained from either normal New Zealand rabbits or rabbits of the same strain made anemic by treatment with APH, was collected directly into either a sucrose-medium A buffer $1$ or into a Tris buffer medium containing $\beta$-mercaptoethanol. $2$ The bone marrow was homogenized at $2^\circ$C for one min with a motor-driven Teflon pestle and centrifuged at 15,000 $\times$ g for 20 min at $2^\circ$C. The supernatant solution...