Ibid., ref. 11; "Examination of methods of attack on the second case of Fermat's last theorem," these PROCEEDINGS, 40, 732-735 (1954); Selfridge, J. L., C. A. Nicol, and V., "Proof of Fermat's last theorem for all prime exponents less than 4002," these PROCEEDINGS, 41, 970-973 (1955).


11 In one of the author's previously published papers (F. L. T. paper no. 3 in ref. 1), the author considered the equation,

\[(x^l + y^l)/(x + y) = u^l \quad (A)\]

and examined the possibilities of this having solutions with \(x, y,\) and \(u\) co-prime integers without employing the relation

\[x + y = v^l. \quad (B)\]

Of course if both of these relations can be satisfied in nonzero integers, then relation (1a) of the present paper is satisfied. In our paper just referred to, it was shown that when \(l = 3,\) the relation (A) has solutions when \(y \equiv 0 \pmod{12}.)\) However, it has turned out in various investigations when \(x, y,\) and \(z\) are each prime to \(l,\) that when criteria have been obtained for the solution of (1a), the case \(l = 3\) would be an exception to the criteria found. There is a very good reason. Relation (A) also has solutions with \(x\) and \(y\) prime to \(l\) and \(|xyu| \neq 1.\) In fact,

\[x = -c_0^4 + 3c_0^3c_1 - c_1^4 \quad (C)\]

\[y = -c_0^4 + 3c_0c_1^2 - c_1^4\]

in (A). As a particular example, it may be verified that with \(c_0 = 4, c_1 = 1,\) we obtain

\[(17)^4 - 53.17 + (53)^3 = 13^4.\]

This result might be known. (Cf. references given by Vandiver, these PROCEEDINGS, 46, 550 (1960)).


THE VIRIAL THEOREM IN GENERAL RELATIVITY IN THE POST-NEWTONIAN APPROXIMATION*

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1. Introduction.—It is a well-known theorem in classical mechanics that, in a system consisting of point masses interacting with each other according to Newton's law of gravitation, the kinetic and the potential energies of the system, when averaged over a sufficiently long interval of time, tend to be in the ratio 2:-1, provided the motion is such that the particles remain, forever, in a finite part of the phase space. The origin of this theorem is the relation

\[\frac{1}{2} \frac{d^2}{dt^2} \sum_m m |\dot{\mathbf{x}}|^2 = \sum_m m u^2 - \frac{1}{2} \sum_m \sum_{m'} G_m \sum_{m'} m m' \frac{m m'}{||\mathbf{x} - \mathbf{x}'||}\]

(1)

which follows readily from the Newtonian equations of motion,
In equation (1), \( x \) denotes the coordinate of a typical particle, \( m \) its mass, \( u \) the magnitude of its velocity \( u \), and \( G \) is the constant of gravitation. Further, in equation (1) and in the sequel, the single summations (over \( m \)) are over all particles in the system and the double summations (over \( m \) and \( m' \)) are over all pairs of particles.

A more general tensor form of equation (1) is

\[
\frac{1}{2} \frac{d^2}{dt^2} \sum_m m x_i x_j = \frac{d}{dt} \sum_m m u_i x_j - \sum_m m u_i u_j - \frac{1}{2} G \sum_m \sum_{m'} m m' \frac{(x_i - x_i')(x_j - x_j')}{|x - x'|^3};
\]  

(2)
equation (1) is the contracted version of this equation.

In terms of the tensor-potential

\[
\mathcal{B}_{ij} = G \sum_{m'} \frac{m' (x_i - x_i')(x_j - x_j')}{|x - x'|^3},
\]  

(3)
we can define the potential-energy tensor

\[
\mathcal{W}_{ij} = -\frac{1}{2} \sum_m \mathcal{B}_{ij},
\]  

(4)
and equation (2) can be written in the form

\[
\frac{d}{dt} \sum_m m u_i x_j = 2 \mathcal{I}_{ij} + \mathcal{W}_{ij},
\]  

(5)
where

\[
\mathcal{I}_{ij} = \frac{1}{2} \sum_m m u_i u_j
\]  

(6)
is the energy-momentum tensor.

If the motions are confined, forever, to a finite part of the phase space, then equation (5), when averaged over a sufficiently long interval of time \( T \), leads to the result

\[
2 < \mathcal{I}_{ij} > + < \mathcal{W}_{ij} > = \lim_{T \to \infty} \frac{1}{T} \left[ \sum_m m u_i x_j \right]_0^T = 0,
\]  

(7)
where the angular brackets signify that the quantity enclosed has been averaged over a time \( T \to \infty \).

Extensions of the theorem we have enunciated to continuous distributions of matter which are subject to material pressure, and other forces and fields (such as centrifugal and Coriolis forces and magnetic fields) are known and have been used. In this paper we shall be concerned with the generalization of equations (5) and (7) to general relativity in the framework of the post-Newtonian approximation of Einstein, Infeld, and Hoffmann.

2. The equations of motion in the post-Newtonian approximation; the energy and the momentum integrals.—The theory of Einstein, Infeld, and Hoffmann is set out
in detail by Infeld and Plebanski in their recent book\textsuperscript{4} \textit{Motion and Relativity}; a briefer account of the essential elements of the theory is given by Landau and Lifshitz\textsuperscript{5} in their \textit{Classical Theory of Fields} (chap. 11, § 105).

The principal result of the Einstein-Infeld-Hoffmann theory is the derivation of a Lagrangian \( L \) in terms of which the post-Newtonian equations of motion of a system of point masses can be written down. The Lagrangian in question is (Infeld and Plebanski, \textit{loc. cit.}, p. 112, eq. (3.3.37) and Landau and Lifshitz, \textit{loc. cit.}, p. 372)

\[
L = \frac{1}{2} \sum_m m u_i^2 \left( 1 + \frac{u^2}{4c^2} + \frac{3G}{c^2} \sum_{m'} \frac{m'}{|\mathbf{x} - \mathbf{x}'|} \right)
- \frac{G}{4c^2} \sum_m \sum_{m'} mm' \left[ \frac{7}{|\mathbf{x} - \mathbf{x}'|} + \frac{u_k(x_k - x_k')u_i'(x_i - x_i')}{|\mathbf{x} - \mathbf{x}'|^3} \right]
+ \frac{1}{2} \frac{G}{2c^2} \sum_{m'} \sum_{m''} mm'' - \frac{G^2}{2c^2} \sum_m \sum_{m'} \sum_{m''} \frac{mm'm''}{|\mathbf{x} - \mathbf{x}'| |\mathbf{x} - \mathbf{x}''|},
\]  

(8)

where, in the triple summation in the last term, the only restriction is that neither \( m' \) nor \( m'' \) can be the same as \( m \); in particular, the possible identity of \( m' \) and \( m'' \) is not excluded. Also, in equation (8), \( c \) denotes the velocity of light. An alternative form of \( L \) which is sometimes useful is

\[
L = \frac{1}{2} \sum m u_i^2 \left( 1 + \frac{u^2}{4c^2} + \frac{3G}{c^2} \mathfrak{B} \right) - \mathfrak{B} \frac{1}{2c^2} \sum m \mathfrak{B}^2
- \frac{2G}{c^2} \sum_m \sum_{m'} \frac{mm'u'u'}{|\mathbf{x} - \mathbf{x}'|} - \frac{G}{4c^2} \sum_m \sum_{m'} mm'u_i' \frac{\partial^2}{\partial x_k \partial x_i'} |\mathbf{x} - \mathbf{x}'|,
\]  

(9)

though this form can be misleading when differentiating with respect to the coordinates of a particular particle (if account is not taken of the fact that \( \mathfrak{B} \) and \( \mathfrak{B}^2 \) are, themselves, defined as sums over the different particles and pairs of particles, respectively).

In terms of \( L \), the equations of motion, as usual, are given by

\[
p_i = \frac{\partial L}{\partial u_i} \quad \text{and} \quad \frac{dp_i}{dt} = \frac{\partial L}{\partial x_i}.
\]  

(10)

With \( L \) given by equation (8), we find

\[
p_i = mu_i \left( 1 + \frac{u^2}{2c^2} + \frac{3}{c^2} \mathfrak{B} \right) - \frac{7Gm}{2c^2} \sum_{m'} \frac{m'u_i'}{|\mathbf{x} - \mathbf{x}'|}
- \frac{Gm}{2c^2} \sum_{m'} (x_i - x_i')u_i'(x_i - x_i')
\]  

(11)

and

\[
\frac{\partial L}{\partial x_i} = -Gm \sum_{m'} m' \left[ 1 + \frac{3}{2c^2} (u^2 + u'^2) - \frac{7}{2c^2} u_i' \right] \frac{x_i - x_i'}{|\mathbf{x} - \mathbf{x}'|^3}.
\]
\[ + \frac{3Gm}{2c^2} \sum_{m'} m' \frac{u_k(x_k - x_k') u'_i(x_i - x_i')}{|x - x'|^3} x_i - x_i' \]

\[ - \frac{Gm}{2c^2} \sum_{m'} m' \frac{(u_k u_k' + u_k u_i')(x_k - x_k')}{|x - x'|^3} \]

\[ + \frac{G^2m}{c^2} \sum_{m'} \sum_{m''} m'm'' \left[ \frac{1}{|x - x'|} + \frac{1}{|x' - x''|} \right] x_i - x_i' \]

In equations (11) and (12) the summations over \( m' \) and over \( m'' \) are restricted only by the requirement that the terms which give rise to singularities are to be omitted.

The equations of motion derived from a Lagrangian \( L \), in the manner (10), will naturally allow the energy integral

\[ E = \sum_m p_i u_i - L. \] (13)

Using equations (8), (11), and (12), we find (cf. Infeld and Plebanski, loc. cit., p. 114, eq. [3.4.3])

\[ E = \sum_m m u_i \left( \frac{1}{2} + \frac{3}{8c^2} u^2 + \frac{3}{2c^2} \mathcal{B} \right) + \mathcal{B} + \frac{1}{2c^2} \sum_m m \mathcal{B}^2 \]

\[ - \frac{7G}{4c^2} \sum_m \sum_{m'} mm' u \cdot u' \cdot \mathcal{B} - \frac{G}{4c^2} \sum_m \sum_{m'} mm' \frac{u_k(x_k - x_k') u'_i(x_i - x_i')}{|x - x'|^3} \] (14)

The constancy of the total momentum,

\[ P_i = \sum_m p_i = \sum_m m u_i \left( 1 + \frac{u^2}{2c^2} - \frac{\mathcal{B}}{2c^2} \right) - \frac{G}{2c^2} \sum_{m'} \sum_{m''} mm' \frac{(x_i - x_i') u'_i(x_i - x_i')}{|x - x'|^3} \]

\[ = \sum_m m u_i \left[ \frac{1}{2c^2} (u^2 - \mathcal{B}) \right] - \frac{1}{2c^2} \sum_m m u_i \mathcal{B}_i, \] (15)

follows from the manifest validity of

\[ \sum_m \frac{\partial L}{\partial x_i} = 0. \] (16)

Since (as can be readily verified)

\[ P_i = \frac{d}{dt} \sum_m m x_i \left[ 1 + \frac{1}{2c^2} (u^2 - \mathcal{B}) \right], \] (17)

the uniform motion of the center of gravitational mass, defined by

\[ \text{Constant} \times X_i = \sum m x_i \left[ 1 + \frac{1}{2c^2} (u^2 - \mathcal{B}) \right], \] (18)

follows at the same time (cf. Infeld and Plebanski, loc. cit., p. 116, eqs. [3.4.9] and [3.4.10]).
3. The conservation of the angular momentum and the virial theorem.—Consider the virial

\[ V_{i;j} = \sum_m p_i x_j \]

\[ = \sum_m m u_i x_j \left( 1 + \frac{1}{2c^2} u^2 + \frac{3}{c^2} \mathcal{V} \right) - \frac{7G}{2c^2} \sum_m \sum_{m'} mm' u_i' x_j \]

\[ - \frac{G}{2c^2} \sum_m \sum_{m'} mm' \frac{u_i'(x_i - x_i')(x_i - x_i')}{|x - x'|^3} \]  

(19)

The time derivative of the virial is given by

\[ \frac{dV_{i;j}}{dt} = \sum_m p_i u_j + \sum_m \frac{\partial L}{\partial x_i} x_j = Q_{ij} \text{ (say)}. \]  

(20)

On evaluating \( Q_{ij} \) with the aid of equations (11) and (12), we find, after some rearrangements,

\[ Q_{ij} = \sum_m m u_i u_j \left( 1 + \frac{1}{2c^2} u^2 + \frac{3}{c^2} \mathcal{V} \right) + \mathcal{V}_{ij} - \frac{1}{c^2} \sum_m \left( \frac{3}{2} u^2 - \mathcal{V} \right) \mathcal{V}_{ij} \]

\[ - \frac{7G}{2c^2} \sum_m \sum_{m'} \frac{mm'}{|x - x'|} \left[ u_i u_j' - \frac{1}{2} \frac{(x_i - x_i')(x_j - x_j')}{|x - x'|^2} u \cdot u' \right] \]

\[ - \frac{G}{2c^2} \sum_m \sum_{m'} mm' \frac{u_i'(x_i - x_i')(x_i - x_i')}{|x - x'|^3} \left[ u_j(x_i - x_i') + u_i(x_j - x_j') \right. \]

\[ \left. - \frac{3}{2} \frac{(x_i - x_i')(x_j - x_j')}{|x - x'|^2} u_i(x_i - x_i') \right]. \]  

(21)

The tensor \( Q_{ij} \) is manifestly symmetric in \( i \) and \( j \). Hence,

\[ \frac{d}{dt} \sum_m (p_i x_j - p_j x_i) = 0; \]  

(22)

and this equation represents no more than the conservation of angular momentum which obtains in this post-Newtonian approximation (cf. Infeld and Plebanski, loc. cit., eq. [3.4.8]).

On the other hand, since \( Q_{ij} \) is the time derivative of the virial \( V_{i;j} \), it follows that its average, taken over a sufficiently long interval of time, will vanish, provided the motions are such that they are always confined to a finite part of the phase space; thus, on this last assumption

\[ \langle Q_{ij} \rangle = 0. \]  

(23)

Equation (23) provides the required generalization of equation (7) to the post-Newtonian approximation; as such, it represents the statement of the virial theorem in that approximation.

The contracted version of equation (23) is particularly simple; for
\[ Q_{ii} = \sum_m m u^2 \left( 1 + \frac{1}{2c^2} u^2 + \frac{3}{2c^2} \mathcal{B} \right) + \mathcal{B} + \frac{1}{c^2} \sum_m m \mathcal{B}^2 \]

\[ - \frac{7G}{4c^2} \sum_m \sum_{m'} \frac{m' \mathbf{u} \cdot \mathbf{u}'}{|x - x'|} - \frac{G}{4c^2} \sum_m \sum_{m'} m' \mathbf{u}_k(x_k - x_k')(x_i - x_i') \frac{1}{|x - x'|^3}; \]

and, by comparison with the energy integral (14), we may now write

\[ \langle Q_{ii} \rangle = E + \frac{1}{2} \sum_m m u^2 > + \frac{1}{2c^2} \sum m \left( \frac{1}{4} u^4 + \mathcal{B}^2 \right) = 0. \] (25)

Equation (25) differs from the Newtonian form only by the addition of the last term.

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† On leave of absence from the Astronomical Department, University of Thessaloniki, Greece.
2 The tensors \( \mathcal{Q}_{ij} \) and \( \mathcal{R}_{ij} \) were defined in the references given in the preceding note; their properties are further studied in Chandrasekhar, S., and N. R. Lebovitz, Ap. J., 135, 248 (1962); Ap. J., 136, 1032 (1962).
6 It should be noted that this result depends on evaluating the time derivative of \( (u^2 - \mathcal{B})/2c^2 \) in the Newtonian approximation; this is clearly justified in the present framework.

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**IMMUNOASSAY OF BOVINE AND HUMAN PARATHYROID HORMONE**

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**BRONX VETERANS HOSPITAL AND THE NATIONAL INSTITUTES OF HEALTH**

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Certain protein hormones in plasma have been assayed by sensitive immunologic techniques that exploit the ability of endogenous plasma hormone to inhibit, competitively, the binding of \(^{125}\text{I}-\text{labeled hormone to specific antibody. Originally employed for measurement of insulin,}^1 - ^3 \) this method of immunoassay has also been applied to the determination of glucagon\(^4 \) and growth hormone\(^5 - ^7 \) in plasma. Since there has not heretofore been available any method for the assay of the minute quantities of parathyroid hormone (Pth) in human blood, the adaptation of immunologic techniques to the assay of Pth in the \( \mu g-\mu g \) range is herein reported.