ON CURVATURE AND EULER-POINCARE CHARACTERISTIC

BY R. L. BISHOP AND S. I. GOLDBERG

UNIVERSITY OF ILLINOIS

Communicated by S. Bochner, April 5, 1963

1. Introduction.—Perhaps the most significant aspect of differential geometry is that which deals with the relationship between the curvature properties of a Riemannian manifold $M$ and its topological structure. One of the beautiful results in this connection is the (generalized) Gauss-Bonnet theorem which relates the curvature of compact and oriented even dimensional manifolds with an important topological invariant, viz., the Euler-Poincaré characteristic $\chi(M)$ of $M$. In the 2-dimensional case, the sign of the Gaussian curvature determines the sign of $\chi(M)$. Moreover, if the Gaussian curvature vanishes identically, so does $\chi(M)$. In higher dimensions, the Gauss-Bonnet formula is not so simple, and one is led to the following important question:

Question: Does a compact and oriented Riemannian manifold of even dimension $n = 2m$ whose sectional curvatures are all nonnegative have nonnegative Euler-Poincaré characteristic, and if the sectional curvatures are nonpositive, is $(-1)^m\chi(M) \geq 0$?

H. Samelson$^7$ has verified this for homogeneous spaces of compact Lie groups with the bi-invariant metric. Unfortunately, however, a proof employing the Gauss-Bonnet formula is lacking. An examination of the Gauss-Bonnet integrand at one point of $M$ leads one to an extremely difficult algebraic problem which has been resolved in dimension 4 by J. Milnor:

**Theorem 1.** A compact and oriented Riemannian manifold of dimension 4 whose sectional curvatures are all nonnegative or all nonpositive has nonnegative Euler-Poincaré characteristic. If the sectional curvatures are always positive or always negative, the Euler-Poincaré characteristic is positive.

A subsequent proof was provided by S.-S. Chern.$^4$ A new and perhaps clearer version indicating some promise for the higher dimensional cases is presented. This proof is not essentially different from the one given in reference 4. An application of our method yields$^{10}$

**Theorem 2.** In order that a 4-dimensional compact and orientable manifold $M$ carry an Einstein metric, i.e., a Riemannian metric of constant Ricci or mean curvature $R$, it is necessary that its Euler-Poincaré characteristic be nonnegative.

This generalizes a result due to H. Guggenheimer.$^5$

Applying a standard minimization technique, one obtains

**Corollary.** If $V$ is the volume of $M$,

$$\chi(M) \geq \frac{VR^2}{12\pi^2},$$

equality holding, if and only if $M$ has constant curvature.

Theorem 2 is improved by relaxing the restriction on the Ricci curvature.

As a first step to the general case, it is natural to consider manifolds with specific curvature properties. A large class of such spaces is afforded by those complex manifolds having the Kaehler property. For this reason, the curvature properties of Kaehler manifolds are examined. We are especially interested in the relationship
between the holomorphic and nonholomorphic sectional curvatures. In particular, sharper bounds on curvature than those given by M. Berger\(^1\) are obtained. Milnor's result is also partially improved by restricting the hypothesis to the holomorphic sectional curvatures. Indeed, the following theorem is proved:

**Theorem 3.** A compact Kaehler manifold of dimension 4 whose holomorphic sectional curvatures are all nonnegative or all nonpositive has nonnegative Euler-Poincaré characteristic. If the holomorphic sectional curvatures are always positive or always negative, the Euler-Poincaré characteristic is positive.

This verifies geometrically, at least for \( n = 4 \), the main result (Theorem 8) in a paper due to S. Bohner.\(^3\)

Assume that the metric of the Kaehler manifold \( M \) has been normalized so that every holomorphic sectional curvature \( H(X) \) satisfies \( \lambda \leq H(X) \leq 1 \). The manifold is then said to be \( \lambda \)-holomorphically pinched.\(^1,\,5\)

By applying an extension of the technique used to obtain Theorem 1, together with a standard maximization method, an upper bound for \( \chi(M) \) is obtained in terms of the volume and the maximum absolute value of holomorphic curvature of \( M \).

**Theorem 4.** Let \( M \) be a compact 4-dimensional Kaehler manifold, \( L \) the maximum absolute value of holomorphic curvature, and \((1 - \lambda)L\) the variation (maximum minus minimum) of holomorphic curvature. Then,

\[
\chi(M) \leq \frac{1}{8\pi^3} (3\lambda^2 - 4\lambda + 4)L^2V.
\]

More important, an upper bound may be obtained in terms of curvature alone when holomorphic curvature is strictly positive. The technique employed to yield this bound also gives a known bound for the diameter of \( M \).\(^1,\,8\) Indeed, a bound \( B \) on the Jacobian of the exponential map is first obtained by employing various facts about the exponential map, Jacobi fields, and 2nd variation of arc length. The bound on volume is then derived by integrating \( B \) on the interior of a sphere of radius \( \pi/\sqrt{\lambda} \) in the tangent space.

In particular, if \( \dim M = n \leq 10 \),

\[
V \leq \frac{2\pi^m}{(m - 1)!\lambda^m} \int_0^\pi x^{n-2} \sin x \exp \left[ -\frac{(n - 2)(3\lambda - 1)}{48\lambda} x^2 \right] dx,
\]

where \( m = \dim_{\mathbb{C}} M \). Applying Theorem 4, we find an upper bound for the Euler-Poincaré characteristic of a complete 4-dimensional \( \lambda \)-holomorphically pinched Kaehler manifold with \( \lambda > 0 \),

\[
\chi(M) \leq \frac{3\lambda^2 - 4\lambda + 4}{4\lambda^2} \int_0^\pi x^2 \sin x \exp \left( -\frac{3\lambda - 1}{24\lambda} x^2 \right) dx.
\]

These bounds may be substantially improved by employing the methods of the calculus of variations.

The technique for estimating volume may be applied to the Riemannian case, obtaining an inequality \( V \leq \sigma(n, D, R/(n - 1)) \), where \( \sigma(n, D, K) \) is the volume of a ball of radius \( D \) in the \( n \)-dimensional space-form of curvature \( K \), \( D \) is the diameter of \( M \), and \( R \) is a lower bound for the Ricci curvature of \( M \). This generalizes the
principal result of M. Berger on -pinched manifolds, the latter being obtained by letting \( R = (n - 1)\delta \) and \( D = \pi/\sqrt{\delta} \) in our inequality. The improvement comes from generalizing Rauch’s theorem so as to estimate directly lengths of Grassman \((n - 1)\)-vectors mapped by \( \exp \) rather than from using Rauch’s estimate of lengths of vectors to estimate lengths of \((n - 1)\)-vectors, as M. Berger does.

Let \( M \) be a Kaehler manifold with almost complex structure tensor \( J \). Let \( G_{n,p}^2 \) denote the Grassman manifold of 2-dimensional subspaces of \( T_p \) — the tangent space at \( P \in M \) and consider the subset

\[
H_{n,p}^2 = \{ \sigma \in G_{n,p}^2 | J \sigma = \sigma \text{ or } J \sigma \perp \sigma \}.
\]

The plane section \( \sigma \) is called holomorphic if \( J \sigma = \sigma \), and anti-holomorphic if \( J \sigma \perp \sigma \) i.e., if it has a basis \( X, Y \) where \( X \) is perpendicular to both \( Y \) and \( JY \). Let \( R(\sigma) \) denote the curvature transformation associated with an orthonormal basis of \( \sigma \) and \( K(\sigma) \) the sectional curvature at \( \sigma \in G_{n,p}^2 \).

A Kaehler manifold is said to have the property \((P)\) if, at each point of \( M \), there exists an orthonormal holomorphic basis \( \{X_\alpha\} \) of the tangent space with respect to which

\[
(R_\alpha(\sigma))^2 = -(K(\sigma))^2 I
\]

for all sections \( \sigma = \sigma(X_\alpha, X_\beta) \) where \( R_\alpha(\sigma) \) denotes the restriction of \( R(\sigma) \) to the section \( \sigma \), and \( I \) is the identity transformation (in other words, in the case where \( K(\sigma) \neq 0 \), \( R_\alpha(\sigma) \) defines a complex structure on \( \sigma \).

The property \((P)\) is preserved under Kaehlerian products. In particular, products of complex projective spaces satisfy this property.

We shall prove

**Theorem 5.** Let \( M \) be a 6-dimensional compact Kaehler manifold having the property \((P)\). If for all \( \sigma = \sigma(X_\alpha, X_\beta) \), \( K(\sigma) \geq 0 \), then \( \chi(M) \geq 0 \), and if \( K(\sigma) \leq 0 \), \( \chi(M) \leq 0 \).

A similar statement is valid for manifolds of dimension \( 4k \). A Kaehler manifold possessing the property \((P)\) for all \( \sigma \in H_{n,p}^2 \) has constant holomorphic curvature.

A procedure is outlined by which a meaningful formula for the Gauss-Bonnet integrand can be found when \( n = 6 \). The formula obtained is then used to yield

**Theorem 6.** A \( \lambda \)-holomorphically pinched 6-dimensional complete Kaehler manifold, \( \lambda \geq 2 - 2^{2/3}(\sim 0.42) \), having the property \((P)\), has positive Euler-Poincaré characteristic.

We note that the Ricci curvature is positive definite for this value of \( \lambda \), this being a consequence of a formula obtained relating curvature with holomorphic curvature.

An obvious modification gives negative characteristic when the holomorphic curvatures lie between \(-1\) and \(-2 + 2^{2/3}\).

An example based on the formula employed to give Theorem 6 shows that for a compact Kaehler manifold of dimension \( \geq 6 \), it is not possible to prove, using only the algebra of the curvature tensor at a point, that nonnegative holomorphic curvature yields a nonnegative Gauss-Bonnet integrand.

In fact, we are strongly of the opinion that the Question cannot be resolved in this manner.

T. Frankel has conjectured that the compact Kaehler manifolds of strictly positive curvature are topologically, and even analytically, the same as the complex
projective spaces. A. Andreotti and Frankel have already established this in dimension 4. In dimension 6, it is not yet known whether a compact Kaehler manifold of positive curvature is homologically complex projective space.

Complete proofs and details will be presented elsewhere.

* The research of this author was supported by the Air Force Office of Scientific Research.
10 Theorem 2, with the exception of the Corollary, was announced by Berger in an invited address at the International Congress of Mathematicians, Stockholm, 1962.

---

**SOME REMARKS ON AUTOMORPHISMS, ANALYTIC BUNDLES, AND EMBEDDINGS OF COMPLEX ALGEBRAIC VARIETIES**

**By Phillip A. Griffiths**

84 GYPSY LANE, BERKELEY, CALIFORNIA

*Communicated by D. C. Spencer, April 2, 1963*

1. Let $X$ be a compact, connected complex manifold (nonsingular), let $A(X)$ be the complex Lie group of analytic automorphisms of $X$, and $A^0(X) = A$ the identity component of $A(X)$. Furthermore, let $E \to \mathcal{E} \to X$ be an analytic vector bundle arising from an analytic principal bundle $G \to P \to X$ by a linear action of the complex Lie group $G$ on a complex vector space $E$. Let $\mathcal{E}$ be the sheaf of germs of holomorphic cross sections of $E$; denote by $\Theta$ the sheaf associated to the trivial bundle, and by $\Theta$ the sheaf associated to the holomorphic tangent bundle $T_X$ of $X$. Associated to $G \to P \to X$, we have the Atiyah sequence

\[ O \to \mathcal{E} \to \mathcal{Q} \to T_X \to O \]

and the corresponding sheaf sequence

\[ O \to \mathcal{E} \to \mathcal{Q} \to \Theta \to O. \tag{1} \]

We record some interpretations of the groups arising from the exact cohomology sequence of (1).

(i) $H^0(X, \mathcal{E})$ represents the infinitesimal bundle automorphisms of $P$ which project to the trivial automorphism of $X$.

(ii) $H^0(X, \mathcal{Q})$ gives the infinitesimal bundle automorphisms of $P$.

(iii) $H^0(X, O) \cong \mathfrak{a}$ represents the complex Lie algebra $\mathfrak{a}$ of $A$. 