will occur with lower frequencies beginning with the fundamental frequency of rotation.

The most interesting features of the materials here described have so far not even been touched upon. The high dielectric constant and the enormous anisotropy could lend themselves to various sorts of experiments not feasible previously. The high dielectric constant would render observable some small effects that occur in the theory of the ponderomotive forces on material dielectrics. Similarly, the great anisotropy should allow the study of optical effects which are present in crystals only to a minute extent due to the small difference between the dielectric constants.

To Professor v. Hippel we want to extend thanks for his help with the measurement of the dielectric constant parallel to the thickness of the samples. Dr. M. H. Johnson, now at Aeroneutronics, and Dr. R. W. Wright, now in charge of the study of absorptive materials for microwaves at NRL, have provided valuable collaboration in the construction and study of the new materials.

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ON THE NUCLEI OF A SIMPLE JORDAN ALGEBRA*

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The purpose of this note is to provide a reasonably direct proof\(^1\) of the following result.

**Theorem.** The nuclei of any simple Jordan algebra of characteristic not two are all equal to its center.

We begin our proof with the following trivial remarks. The *right nucleus* of any algebra \( \mathfrak{A} \) is the set \( \mathfrak{A}_\rho \) of all elements \( a_\rho \) of \( \mathfrak{A} \) such that \((xy)a_\rho = x(ya_\rho)\) for every \( x \) and \( y \) of \( \mathfrak{A} \). The *left nucleus* \( \mathfrak{A}_\lambda \) is the set of all \( a_\lambda \) such that \( a_\lambda(xy) = (a_\lambda x)y \). The *middle nucleus* is the set \( \mathfrak{A}_\mu \) of all \( a_\mu \) such that \((xa_\mu)y = x(a_\mu y)\). When \( \mathfrak{A} \) is commutative, \( \mathfrak{A}_\rho \) and \( \mathfrak{A}_\lambda \) are clearly identical. However, \((xy)a = x(ya)\) implies that \( x(ay) = (xy)a = (yx)a = y(xa) = (xa)y \). Thus, \( \mathfrak{A}_\rho \subseteq \mathfrak{A}_\mu \). Since the center \( \mathfrak{R} \) of \( \mathfrak{A} \) is the intersection of the nuclei in the commutative case, we actually have

\[
\mathfrak{R} = \mathfrak{A}_\rho \subseteq \mathfrak{A}_\mu, \quad (1)
\]

for all commutative algebras \( \mathfrak{A} \).

Let us use the standard notation \( xy = xR_y = yx \), where \( R_y \) is a linear transformation on \( \mathfrak{A} \) called a *right multiplication*. Then it is well known\(^2\) that the identity

\[
R_x(yz-(xy)z) = (R_xR_z - R_zR_x)R_y - R_y(R_xR_z - R_zR_x) \quad (2)
\]

holds. If \( y = a \) is in \( \mathfrak{A}_\mu = \mathfrak{R} \), the left member of (2) vanishes and (2) becomes

\[
(R_xR_y - R_yR_x)a = R_x(R_xR_y - R_yR_x) \quad (3)
\]

\( (x, y \in \mathfrak{A}, a \in \mathfrak{R}) \).
Write

\[ b = x(ya) - (xy)a \]  
(4)

and let \( B \) be the subspace of \( \mathfrak{A} \) of all finite sums of elements of the form \( x(ya) - (xy)a \) as in (4). Then

\[ b' = [(wy)a]x - [(wy)x]a = x[(wy)a] - [x(wy)]a \]  
(5)

is in \( B \), and so are

\[ b'' = [(wx)a]y - [(wx)y]a, \quad b''' = [(xy)a]w - [(xy)w]a. \]  
(6)

We now write

\[ R_b = R_{(yw)-(xy)a} = R_x(R_yR_z - R_zR_y) + (R_yR_z - R_zR_y)R_y, \]  
(7)

and compute

\[ wb = b' + [(wx)a - (wa)x]y = b' + q. \]  
(8)

By (3) we can compute \( (R_zR_y - R_yR_z)R_y = R_zR_yR_y - R_yR_zR_y - R_zR_yR_z + R_yR_zR_z \) and this is in \( B \). Thus, \( q = [(wx)a]y - [(wy)x]a + [(wy)x]a - [(wa)y]x = b'' + [(wy)x]a - [(wa)y]x. \) However, \( [(wy)x]a - [(wa)y]x = [(y(w)x)a - [(ya)x]w, since (wa)y = w(ay). \) Thus, \( q - b'' \) is unaltered by the interchange of \( w \) and \( y \), and so \( q - b'' = [(yx)a - [(ya)x]w - (yb') - b''' = -wb - b''' \), \( q = -wb + b' - b''' \), and we have proved that

\[ 2wb = b' + b'' - b'''. \]  
(9)

This proves that \( B \) is an ideal of \( \mathfrak{A} \). Hence, \( B = 0 \) and \( \mathfrak{A} = \mathfrak{A} \) as desired, or \( B = \mathfrak{A} \).

At this point we do need to introduce some structure theory. We first extend \( \mathfrak{A} \) to its algebraic closure \( \Omega \) and so obtain \( \mathfrak{A}_\Omega = \mathfrak{A}_\Omega \). When \( \mathfrak{A} \) is special, there is a trace function \( \delta(x) \) such that \( \delta(b) = 0 \) for every \( b \) in \( \mathfrak{B} \). Also, if \( u \) is a primitive idempotent of \( \mathfrak{A} \), it is true that \( \delta(u) = 1, 2, \) or \( 4 \neq 0 \), when \( \mathfrak{A} \) has characteristic not two. Hence, \( u \) cannot be in \( \mathfrak{A}_\Omega \) and so \( \mathfrak{A} \neq \mathfrak{B} \). When \( \mathfrak{A} \) is exceptional and the characteristic of \( \mathfrak{A} \) is not 3, we use the trace function \( \delta(x) \) equal to the trace of the matrix of \( R_z \). By (2) we see that \( \delta(b) = 0 \) for every \( b \) in \( \mathfrak{B} \). But \( \delta(e) \) is the dimension of \( \mathfrak{A} \) and this is 27. When \( \mathfrak{A} \) does have characteristic three, we can compute the middle nucleus of \( \mathfrak{A}_\Omega \) directly to obtain our result.

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* This result appears in a paper by Robert H. Oehmke and Reuben Sandler entitled "The collineation groups of division ring planes I: Jordan algebras." Their proof was made by using the structure theory for \( \mathfrak{A} \) algebraically closed and direct computation.


* Indeed we have shown that, if \( \mathfrak{U} \) is a simple special Jordan algebra over a center \( \mathfrak{A} \), and \( \Omega \) is the algebraic closure of \( \mathfrak{A} \), the trace of the elements of \( \mathfrak{U} \) as matrices satisfies the relation \( \delta(x\mathfrak{U}) = \delta(xyz) \). For proof, see the argument on page 410 of the author's 1954 Annals paper entitled "The structure of right alternative algebras."