ON THE ABSOLUTE STABILITY OF SAMPLED-DATA CONTROL SYSTEMS*

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1. Let \( \varphi = \varphi(\sigma) \) be a continuous real scalar function of the real variable \( \sigma \) with continuous and bounded derivative and such that
\[
\varphi(0) = 0 \quad 0 < \sigma \varphi(\sigma) \leq \sigma^k, \quad k < +\infty.
\]

Consider the sampled-data control system \( \Omega \) which consists of a linear and time invariant part and the nonlinearity \( \varphi = \varphi(\sigma) \). Such a system will be represented by the difference equation
\[
x_{h+1} = Ax_h - a\varphi(\sigma_h)
\]
\[
\sigma_h = 2b'x_h, \quad h = 0, 1, \ldots
\]
where \( A \) is a real \( n \times n \) matrix, and \( a \) and \( b \) are real vectors.

We shall assume that the matrix \( A \) is stable, that is, that eigenvalues \( \gamma_i \) satisfy
\[
|\gamma_i| < 1, \quad i = 1, 2, \ldots, n,
\]
that the linear part of \( \Omega \) is completely controllable\(^1\) (the vectors \( a, Aa, \ldots, A^{n-1}a \) are linearly independent) and completely observable\(^1\) (the vectors \( b, A'b, \ldots, (A')^{n-1}b \) are linearly independent), and also that the system (2) is asymptotically stable for all linear \( \varphi(\sigma) = \sigma\mu \) in the class (1). Thus, the system (2) has only one equilibrium point: \( x_h = x_{h+1} = 0 \).

2. One wants to derive a sufficient condition for the existence of a Liapunov function of the type
\[
v = x'Hx + \beta \int_0^\sigma \varphi(s)ds \quad H > 0
\]
which proves global asymptotical stability for the system (2) with respect to all the nonlinearities (1) (absolute stability). Thus, we look for a solution of the Lur'e problem\(^{2,3}\) for (2).

The \( v \)-difference of (4) along the solutions of (1) is:
\[
\Delta v = x_h'[A'H - H]x_h - 2\varphi(\sigma_h)x_h'A'Ha + \varphi^2a'Ha + \beta \int_{\sigma_h}^{\sigma_{h+1}} \varphi(s)ds.
\]

By the mean value theorem, since \( \varphi(\sigma) \) is continuous with continuous and bounded derivative, we have:\(^6\)
\[
\beta \int_{\sigma_h}^{\sigma_{h+1}} \varphi(s)ds \leq \beta \varphi(\sigma_h)[\sigma_{h+1} - \sigma_h] + \frac{\mu}{2} \beta \text{sign} \beta[\sigma_{h+1} - \sigma_h]^2
\]
where
\[
\mu > \frac{d\varphi}{d\sigma}|_{\sigma=0} > 0.
\]

By adding to the scalar function (5) the identity
\[
-\alpha[\varphi(\sigma_h)\sigma_h - 2\varphi(\sigma_h)x_h'b] = 0,
\]
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and taking into account the inequality (6), one obtains:
\[ \Delta v \leq x_h'[A'H A - H + rr']x_h - 2\varphi x_h'[A'H a - ab - \beta d] \]
\[ - \varphi^2[\alpha \frac{1}{k} + 2\beta \xi - a'H a] - \alpha \varphi[\sigma_n \frac{1}{k} \varphi] \]  
(9)

where \( r, d, \) and \( \xi \) are defined, respectively, by:
\[ r = \sqrt{2\mu|\beta|} (A - I)'b \]  
(10)
\[ d = -(2\mu b'a \text{ sign } \beta - 1) (A - I)'b \]  
(11)
\[ \xi = a'b - \mu(a'b)^2 \text{ sign } \beta. \]  
(12)

The expression \( \Delta v \) will be negative definite along the solutions of (2) if there exists a real scalar \( \gamma \), a real vector \( q \), and a real matrix \( H = H' \) such that
\[ A'H A - H + rr' + qq' = 0, \]
\[ A'H a - ab - \beta d = \gamma q, \]
\[ 2\beta \xi + \frac{1}{k} - a'H a = \gamma^2 > 0. \]  
(13)

Then the expression (9) becomes:
\[ \Delta v \leq -[q'x + \gamma \varphi]^2 - \alpha \varphi[\sigma_n \frac{1}{k} \varphi]. \]  
(14)

By using the same procedure as in reference 5, one can show that:

**Lemma.** Necessary and sufficient conditions for the existence of \( \gamma, q, H, \) as above are
\[ \alpha \frac{1}{k} + 2\beta \xi + 2 \text{ Re} \{(ab + \beta d)'(Iz - A)^{-1}a \} - |r'(Iz - A)^{-1}a|^2 \geq 0 \]  
(15)
\[ \alpha \frac{1}{k} + 2\beta \xi > 0 \]  
(16)

for all \( z = e^{i\omega} \) and some real numbers \( \alpha \) and \( \beta \).

By some algebraic manipulations one may show that (15) and (16) can be written in the following simpler form:
\[ (\gamma + \beta) \frac{1}{k} + \text{ Re}(\gamma + \beta z)W(z) - \frac{\mu|\beta|}{2} |(z - 1)W(z)|^2 \geq 0, \]  
(17)
\[ \gamma = \alpha - \beta \]  
(18)

where
\[ W(z) = 2b'(Iz - A)^{-1}a \]  
(19)
is the open-loop transfer function of the linear part of (2), and
\[ (\gamma + \beta) \frac{1}{k} + 2\beta \xi > 0. \]  
(20)
Thus,

**Theorem.** If the conditions (17) and (20) are satisfied for all \( z \) such that \( |z| = 1 \) and for real numbers \( \gamma, \beta \) such that

\[
\gamma + \beta \geq 0
\]

(21)

then the system (2) is absolutely stable.

**Proof:** Let us show that \( \Delta v \) is negative definite along the solutions of (2). When \( \gamma + \beta = 0 \) or \( \varphi(\sigma) = k\sigma \), the inequality (14) becomes

\[
\Delta v \leq - [q'x - \gamma\varphi]^2.
\]

(22)

From (6) one can see that in (22) the equality sign may occur only if \( \sigma_{h+1} = \sigma_h \), that is, \( 2b'(x_{h+1} - x_h) = 0 \), that is, (because of c.o. of the system (2)) if \( x_{h+1} = x_h \), that is, on equilibrium points. But the only equilibrium point of (2) is \( x_{h+1} = x_h = 0 \). We show that, owing to the assumptions made on (2), the scalar function (4) is positive definite whatever the sign of \( \beta \). Assume that \( v \) is semidefinite and denote by \( N \) the set on which \( v(x) = 0 \). Then also \( \Delta v(x) = 0 \) on \( N \) and \( N \) is an invariant set of (2), which is possible if and only if \( N \) is the point \( x = 0 \).

Assume that \( v \) is indefinite. The integral \( \int_0^x \varphi(s)ds \) has its maximum for the linear system \( \varphi(\sigma) = k\sigma \) which is, by assumption, asymptotically stable, hence \( v \) must be positive definite.7

The result proved in this Theorem contains as particular cases those of Szegö and Kalman,8 and of Zypkin.8

3. Consider now some critical cases. Assume that one of the eigenvalues, say \( \gamma_k \), of the matrix \( A \) is unity and all the other eigenvalues of \( A \) satisfy (3). Thus, the condition

\[
\gamma_i\gamma_j \neq 1 \quad i,j = 1,2, \ldots, n
\]

(23)

is still satisfied.

Assume that (17), (20), and (21) hold. The Lemma is still valid. Then, by going through the same procedures as in the previous cases, since, owing to (23), the first equation of (13) still defines \( H = H' \), one is still able to construct a scalar function of type (4) such that \( \Delta v \) is negative definite along the solutions of the system (2). By the same arguments as in the previous section, one may then show that \( v \) is positive definite.

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