Let \( E \subset X \) be the finite-dimensional subspace spanned by \( x_1, \ldots, x_m \). Now, for \( y \in Y \) and variable \( x \in E \), \( y(x) \) is a continuous linear functional on \( E \) which we denote by \( y' \). Thus, there is defined in a natural way a function \( f'(x) \) with domain \( (E \cap B) \) and range in \( E^* \), the conjugate space of \( E \). The domain contains and surrounds densely every point of \( E \cap K(D) \), hence every point of \( K(E \cap D) \). It is easy to show that \( f' \) is monotonic and hemicontinuous. The set \( (E \cap D) \) surrounds the origin in \( E \), and \( x \in (E \cap D) \) implies \( \langle x, f'(x) \rangle \geq 0 \). All the hypotheses of Lemma 2 are satisfied, so there exists \( x \in K(E \cap D) \) with \( f'(x) = \theta \). Of course, it does not follow that \( f(x) = \theta \); however, by the monotonicity of \( f' \), \( \langle x_i - x, f'(x_i) - \theta \rangle \geq 0 \) for \( i = 1, \ldots, m \), and from this (5) follows, since all the \( (x_i - x) \) are elements of \( E \). The proof is complete.

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SPECTRAL OPERATORS IN A DIRECT SUM OF HILBERT SPACES*

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The purpose of this communication is to outline methods which yield a fairly complete spectral analysis for operators in a certain noncommutative \( B^* \)-algebra of operators in the direct sum \( \mathfrak{S}^* = \mathfrak{S} + \ldots + \mathfrak{S} \) of a Hilbert space \( \mathfrak{S} \) with itself \( n \) times. Some of the results described herein are intimately related to the elegant work of S. R. Foguel in his study of the algebra \( \mathfrak{A} \) of operators commuting with a normal operator of finite multiplicity.

Let \( \sum \) be a \( \sigma \)-field of subsets of a set \( \mathfrak{S} \) and \( e(\cdot) \) a countably additive self adjoint spectral measure defined on \( \sum \) whose values are projection operators in \( \mathfrak{S} \). Let \( \mathfrak{B} \) be the algebra of all operators \( b \) in \( \mathfrak{S} \) having the form \( b = \int_{\mathfrak{S}} b(\cdot)e(\cdot)d ) \) for some \( e \)-essentially bounded \( \sum \)-measurable function \( b \) on \( \mathfrak{S} \). This correspondence \( b \leftrightarrow b \) is known\(^2\) to be an isometric \( * \)-isomorphism between the algebra \( \mathfrak{B} \) and the algebra \( eB(\mathfrak{S}) \) of \( e \)-essentially bounded functions on \( \mathfrak{S} \). Now consider the algebra of
mappings \( B: [x_1, \ldots, x_n] \rightarrow [y_1, \ldots, y_n] \) in \( \mathbb{S}^n \) having the form \( y_i = \sum_{k=1}^{n} b_{ik}x_k \) where \( b_{ij} \) is in \( \mathfrak{B} \). This is a noncommutative \( B^* \)-algebra \( \mathfrak{B}^n \) of operators in \( \mathbb{S}^n \) which is isometrically *-isomorphic with the algebra \( eB^* (\mathfrak{E}) \) of all \( \Sigma \)-measurable \( e \)-essentially bounded maps \( j \rightarrow ˆB(j) \) of \( \mathfrak{E} \) into the algebra \( B(E^n) \) of bounded linear operators in \( n \)-dimensional unitary space \( E^n \). The isomorphism is given by the integral \( B = \int_\mathfrak{E} ˆB(j)e(dj) \). It follows that the spectrum of \( B \) consists of those \( \lambda \) for which \( e \)-ess sup \( \sigma(B(j)) \) follows, and we denote \( \sigma(B(j)) \) the Borel set \( \sigma \) of complex numbers for which \( e \)-ess sup \( \sigma(B(j)) \) is equal to \( \sigma(B) \).

Now, for a fixed \( j \) in \( \mathfrak{E} \), the operator \( ˆB(j) \) in \( E^n \), being a spectral operator, has a resolution of the identity whose value on the Borel set \( \sigma \) of complex numbers we will denote by \( E(\sigma; ˆB(j)) \). It may be proved that the condition

\[
\sup_{\sigma \in \mathfrak{B}} \text{e-ess sup } \mathbb{E}(\sigma; ˆB(j)) < \infty, \tag{1}
\]

where \( \mathfrak{B} \) is the Borel field in the complex plane, is equivalent to the condition

\[
\text{e-ess sup } \sup_{\sigma \in \mathfrak{B}} \mathbb{E}(\sigma; ˆB(j)) < \infty. \tag{1'}
\]

This condition implies the following Fubini type formula. Let \( \varphi \) be a bounded Borel scalar function on the spectrum \( \sigma(B) \). Then the integral \( \mathbb{E}_\varphi(B) = \int \mathbb{E}(\sigma; ˆB(j)) \) is an \( e \)-essentially bounded \( \Sigma \)-measurable function of \( j \), the integral \( \mathbb{E}_\varphi(B) = \int_\mathfrak{E} \mathbb{E}(\sigma; ˆB(j)) d\varphi(j) \) is a bounded countably additive spectral measure in \( \mathbb{S}^n \), and

\[
\mathbb{E}_\varphi(B) = \int_\mathfrak{E} \mathbb{E}(\sigma; ˆB(j)) d\varphi(j). \tag{2}
\]

From this formula it may be proved that the operator \( B \) in \( \mathbb{S}^n \) is a spectral operator if and only if the condition (1) holds, and that when it is a spectral operator, it is of type \( n - 1 \), i.e., the \( n \)th power of its radical vanishes. The resolution of the identity for \( B \) is the integral

\[
E(\sigma; B) = \int_\mathfrak{E} \mathbb{E}(\sigma; ˆB(j)) e(d\varphi(j)).
\]

From this result, it is clear that every operator in \( \mathfrak{B}^n \) is the strong limit of a sequence of spectral operators, a result proved by Foguel for the algebra \( \mathfrak{A} \) mentioned earlier.

The projections \( E(\lambda; B(j)) \) may be calculated as polynomials in \( B(j) \) by using interpolating polynomials, and an examination of the form of these polynomials shows that if, for \( e \)-almost all \( j \) in \( \mathfrak{E} \) and some constant \( K \), we have \( |\lambda - \mu|^{-1} < K \), where the supremum is taken over all \( \lambda, \mu \) of distinct eigenvalues of \( B(j) \), then \( B \) is a spectral operator of type \( n - 1 \). However, this condition is not necessary.

Let the complex number system be ordered by defining \( w \leq z \) to mean that \( |w| \leq |z| \) and if \( |w| = |z| \), then arg \( w \leq \arg z \). Let \( \mathfrak{E}_i \) be the set of all \( j \) in \( \mathfrak{E} \) for which the spectrum \( \sigma(B(j)) \) consists of \( i \) distinct points, and let \( \lambda_{ij}(j), j = 1, \ldots, i \) be these distinct characteristic numbers arranged so that \( \lambda_{ij}(j) \leq \lambda_{ik}(j) \). The sets \( \mathfrak{E}_i \) are disjoint sets in \( \Sigma \), whose union is \( \mathfrak{E} \), and the function \( \lambda_{ij}(\cdot) \) is a \( \Sigma \)-measurable function on \( \mathfrak{E}_i \). If \( E(\lambda_{ij}(j); B(j)) \) is the value of the resolution of the identity of \( B(j) \) evaluated on the set consisting of the single point \( \lambda_{ij}(j) \), then the condition (1) is also equivalent to the condition

\[
\text{e-ess sup } \mathbb{E}(\lambda_{ij}(j); B(j)) < \infty, \ 1 \leq j \leq i \leq n. \tag{1''}
\]
In terms of these projections, the operational calculus for the spectral operator \( B \) may be put in a form reminiscent of that for a finite matrix. By way of illustration, let \( S \) be the scalar part of \( B \) and let \( \varphi \) be a bounded Borel scalar function on \( \sigma(B) = \sigma(S) \). Then the operator \( \varphi(S) \) in \( \mathcal{S}^n \) is given by the formula \( \varphi(S) = \sum_{i=1}^{n} \sum_{j=1}^{n} \varphi(\lambda_{ij})E_{ij} \), where \( E_{ij} = \int_{\mathbb{R}} E(\lambda_{ij}(t);\hat{B}(t))e(dt) \) is a projection in \( \mathcal{S}^n \) and the elements \( \lambda_{ij} \), \( 1 \leq j \leq i \leq n \), are operators in \( \mathcal{S} \) defined by the equations \( \lambda_{ij} = \int_{\mathbb{R}} \lambda_{ij}(t)e(dt) \).

The above remarks show that the spectral analysis of such operators \( B \) reduces to the algebraic problem of solving a polynomial of degree \( n \). Among the familiar operations of mathematical analysis which come under the preceding discussion, that of convolutions by \( L_1 \) functions is perhaps the most interesting. By taking the Hilbert space \( \mathcal{S} \) to be \( L_2(-\infty, \infty) \), and by taking the operators \( b_{ij} \) in \( L_2(-\infty, \infty) \) to be operators of the form

\[
b_{ijf} = \alpha_{ijf} + g_{ijf}, \quad f \in L_2(-\infty, \infty),
\]

where \( \alpha_{ij} \) is a complex number, \( g_{ij} \) is a function in \( L_1(-\infty, \infty) \), and

\[
g_{ijf} = \int_{-\infty}^{\infty} g_{ij}(s-t)f(t)dt,
\]

it is seen that a complete spectral analysis of the operator \( B = (b_{ij}) \) in \( \mathcal{S}^n \) reduces to the examination of an \( n \times n \) matrix whose elements are given in terms of the Fourier transforms of the functions \( g_{ij} \).

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**BACTERIOPHAGE-INDUCED MUTATION IN ESCHERICHIA COLI**

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Many hereditary traits of bacteria are determined by a special class of genetic elements termed *episomes*\(^{1-4}\). Episomes usually determine nonessential characters not ordinarily represented in the bacterial gene pool. Some episomic elements, e.g., transducing phages and the sex factor of *Escherichia coli* strain K12, are instrumental in providing means for the intercellular transfer of bacterial genes. It has been postulated that episomes may also regulate the biosynthetic activities of bacteria at the genetic level by stimulating, inhibiting, or otherwise modifying the phenotypic expression of specific genes.\(^{1,5}\) Recent discoveries\(^{6-7}\) indicate that episomes can in fact stimulate the activity of bacterial genes under special conditions, but total suppression of normal genetic functions by an episme has not been observed in Eubacteria.

This report describes a novel temperate bacteriophage, designated phage *Mu*,