To every compact Riemannian manifold $M$ there corresponds the sequence $0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ of eigenvalues for the Laplace operator on $M$. It is not known just how much information about $M$ can be extracted from this sequence. This note will show that the sequence does not characterize $M$ completely, by exhibiting two 16-dimensional tori which are distinct as Riemannian manifolds but have the same sequence of eigenvalues.

By a flat torus is meant a Riemannian quotient manifold of the form $R^n/L$, where $L$ is a lattice (= discrete additive subgroup) of rank $n$. Let $L^*$ denote the dual lattice, consisting of all $y \in R^n$ such that $x \cdot y$ is an integer for all $x \in L$. Then each $y \in L^*$ determines an eigenfunction $f(x) = \exp(2\pi i x \cdot y)$ for the Laplace operator on $R^n/L$. The corresponding eigenvalue $\lambda$ is equal to $(2\pi)^2 y \cdot y$. Hence, the number of eigenvalues less than or equal to $(2\pi)^2$ is equal to the number of points of $L^*$ lying within a ball of radius $r$ about the origin.

According to Witt there exist two self-dual lattices $L_1, L_2 \subset R^{16}$ which are distinct, in the sense that no rotation of $R^{16}$ carries $L_1$ to $L_2$, such that each ball about the origin contains exactly as many points of $L_1$ as of $L_2$. It follows that the Riemannian manifolds $R^{16}/L_1$ and $R^{16}/L_2$ are not isometric, but do have the same sequence of eigenvalues.

In an attempt to distinguish $R^{16}/L_1$ from $R^{16}/L_2$ one might consider the eigenvalues of the Hodge-Laplace operator $\Delta = d\delta + \delta d$, applied to the space of differential $p$-forms. However, both manifolds are flat and parallelizable, so the identity

$$\Delta(f dx_1 \wedge \ldots \wedge dx_p) = (\Delta f) dx_1 \wedge \ldots \wedge dx_p$$

shows that one obtains simply the old eigenvalues, each repeated $\binom{16}{p}$ times.
